SEDIMENT TRANSPORT:

AN APPRAISAL OF AVAILABLE METHODS

VOLUME 1 SUMMARY OF EXISTING THEORIES

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TABLE

1. Summary of theoretical methods

NOTATION

a	Constant, 2.45 $Y_{cr}^{\frac{1}{2}}/s^{0.4}$ (Yalin)
a	Exponent in the concentration distribution equation (Toffaleti)
A_*	Constant (Einstein)
A	Value of F_{gr} at initial motion (Ackers, White)
b	Stream breadth, surface width if not otherwise indicated
B_*	Constant (Einstein)
C	Coefficient in the general function (Ackers, White)
C_x	Constant of proportionality in the concentration distribution equation (Toffaleti)
C_{xp}	Concentration as weight per unit volume in a given size fraction, p, in the lower, middle and upper zones of transport
C_{Lp}	Concentration as weight per unit volume in a grain size fraction, p, in the lower zone of transport (Toffaleti)
$\overline{CL}_2$	Constant of grain volume/constant of grain area (Toffaleti)
C_z	Temperature related parameter used in evaluating the middle zone exponent of the concentration distribution
d	Mean depth of flow
d'	Mean depth with respect to the grain
$D, D_{35}, D_{50}, D_{65}, D_{90}$	Sediment diameters
D_*	Sieve size No 80 (US Standard)
D_{gr}	Dimensionless grain size (Ackers, White)
e_b	Bed load transport efficiency

e_s	Suspended load transport efficiency
E	Efficiency of the sediment transport process (Ackers, White)
F_b	Bed factor (Blench)
F_{bo}	Bed factor when there is no movement of bed particles (Blench)
F_{gr}	Sediment mobility (Ackers, White)
Fr	Froude Number
F_s	Side factor (Blench)
g	Acceleration due to gravity
g_{bt}	Bed load transport rate, dry weight per unit width per unit time
g_{st}	Total transport rate, dry weight per unit width per unit time
G	Bed load transport rate, dry weight per unit time
G^2	Reynold's number for solids sheared in fluid
GF_p	Nucleus load, dry weight per unit width per unit time for a given grain size range
I_1, I_2	Integrals (Einstein)
k	Correction factor for models
k_m	Coefficient of bed roughness, grain and form (Strickler)
k_r	Coefficient of particle friction, plane bed (Strickler)
k_s	Grain roughness of the bed
k_{se}	Coefficient of total roughness, grain and form (Strickler)
k_w	Coefficient of side wall roughness (Strickler)
K	Meander slope correction (Blench)
log	Common logarithm (base 10)

ln	Naperian logarithm (base e)
m	Exponent in the general function (Ackers, White)
M	Sediment transport rate, mass per unit width per unit time
n	Transition exponent (Ackers, White)
n	Manning roughness coefficient
p	The proportion by weight of particles of size D_p or, as a suffix, denoting association with a particular size fraction
p_b	Fraction of bed material in a given range of grain sizes
p_B	Fraction of bed load in a given range of grain sizes
p_t	Fraction of the total load in a given range of grain sizes
P, P_p	Factors which indicate the proportion of the bed taking the fluid shear (Kalinske)
q	Water discharge per unit width
q_{bt}	Bed load transport rate, submerged weight per unit width
q_{st}	Total load transport rate, submerged weight per unit width per unit time
Q	Water discharge
Q_s	That portion of Q whose energy is converted into eddying close to the bed
R	Hydraulic radius
R'	Hydraulic radius ascribed to the grain
s	Specific gravity of sediment
s*	Expression, $(Y/Y_{cr} - 1)$ (Yalin)
S_o	Water surface or energy slope

t	A variable
t_g ψ_o	Coefficient of dynamic friction
T	A parameter including variables which are functions of temperature
T_F	Temperature of water in degrees Fahrenheit
v	Flow velocity at a distance y_* above the bed
v_*'	Shear velocity related to grain (Einstein)
v_*	Shear velocity $(\tau/\rho)^{1/2}$
v_{*c}	Critical shear velocity
V	Mean velocity of flow
w	Fall velocity of the sediment particles
X, X_s, X_b	Concentration by weight, ie weight of sediment/weight of water. Subscript s denotes suspended load, b denotes bed load.
χ	Characteristic grain size of a mixture
x	Parameter for transition (smooth to rough)
y	Pressure correction in transition (smooth to rough)
y_*	Distance above the bed
Y	Dimensionless Mobility Number (Shields)
Y_c	Mobility number at threshold conditions
z	Exponent relating to the suspended distribution
z_v	Exponent relating to the velocity distribution
z_p	Exponent relating to sand distribution in the middle zone of transport
α	A coefficient (Ackers, White and Bagnold)
β, β_x	Logarithmic functions
γ	Specific weight of the fluid
ρ	Density of the fluid
ρ_s	Density of solids

ν	Kinematic viscosity
γ_s	Specific weight of grain in fluid, $g(\rho_s-\rho)$
η_o	Constant (Einstein)
ξ	Hiding factor
θ	Correction factor
ω	Stream power per unit boundary area
τ	Tractive shear
τ_o	Tractive shear associated with sediment particle
τ_c	Critical tractive shear to initiate motion
δ	Thickness of the laminar sublayer
δ'	Laminar sublayer with respect to v_*'
ψ	Intensity of shear on a particle
ψ'	Intensity of shear on representative particle
ψ_*	Intensity of shear for individual grain sizes
ϕ	Intensity of transport
ϕ_*	Intensity of transport of individual grain sizes
λ	Friction factor, $8gRS_o/V^2$
Δ	Strip width, Simpson Rule
Σ	Sum of ...

SEDIMENT TRANSPORT:

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VOLUME 1 SUMMARY OF EXISTING THEORIES

INTRODUCTION

In 1972 Ackers and White (Refs 24 and 25) proposed a new sediment transport theory and tested the theory against flume data. This data consisted of about 1000 measurements with particle sizes in the range $0.04 < D(mm) < 5$ and sediment specific gravities in the range $1.07 < s < 2.65$.

Subsequent phases in this investigation have included (i) the acquisition of about 270 field measurements of transport rates from the literature and (ii) the testing of all commonly used sediment transport theories against this extended range of data. These two phases are the subject of the present report. The report also includes a proposed modification to one of the existing stochastic theories which significantly improves its performance.

Volume 1 describes the sediment transport theories and indicates in each case how the theories have been used in the present investigation. Where the theories include graphical solutions, analytical equivalents have been worked out to facilitate the use of a digital computer for the analysis. The limitations of the theories are indicated wherever the original authors have given specific recommendations.

Volume 2 describes and classifies the data used for comparative purposes. It defines the criteria on which the comparison between observed and calculated transport rates are based and presents the results provided by the theories described in Volume 1. The problems of graded sediments are discussed and illustrated using gravel river data. A modification to the Bishop, Simons and Richardson theory is proposed together with suggestions for further refinements which could be introduced at a later date. The performance of the various theories is compared and recommendations are made regarding their usage.

BED LOAD EQUATION OF A SHIELDS (1936)

In 1936 A Shields, Ref (1), gave the following formula for the bed load transport rate:-

$$\frac{\gamma_s}{S_o \gamma^2} \frac{G}{Q} = 10 \frac{\tau - \tau_c}{D \gamma_s} \qquad \ldots(1)$$

Where γ = specific weight of the fluid

γ_s = specific weight of solids in transport

τ = tractive shear

τ_c = critical tractive shear

D = mean diameter defined as D_{50}

Q = water discharge

G = bed load transport rate, dry weight per unit time

S_o = Energy gradient or surface slope

Substituting

$$Q = bdV \quad \text{...(2)}$$

$$G = b\, g_{bt} \quad \text{...(3)}$$

$$S_o = v_*^2/gd \quad \text{...(4)}$$

$$\tau = \rho v_*^2 = g\rho d S_o \quad \text{...(5)}$$

$$g_{bt} = X_b V d \rho g \quad \text{...(6)}$$

$$\gamma_s/\gamma = s-1 \quad \text{...(7)}$$

in equation (1) gives a concentration for the bed load, X_b, as:-

$$X_b = \frac{10\, v_*^4}{(s-1)^2\, Ddg^2}\left(1 - \frac{v_{*c}^2}{v_*^2}\right) \quad \text{...(8)}$$

But defining the mobility number at the threshold conditions as:-

$$Y_C = v_{*c}^2/(gD(s-1)) \quad \text{...(9)}$$

$$X_b = \frac{10\, v_*^4}{(s-1)^2\, Ddg^2}\left(1 - \frac{Y_C}{v_*^2/gD(s-1)}\right) \quad \text{...(10)}$$

This equation is claimed to be valid at low transport rates and under conditions where bed formations are relatively flat. Knowing v, v_*, d, D, s and g sediment concentrations are calculated as follows:-

1. D_{gr} is computed from equation (161), see page 40

2. Y_{cr} is calculated from equation (A4), see Appendix 1, or from the original graphs of A. Shields (Ref 1).

3. Substitution in equation (10) yields X_b. Equation (10) is, of course, non-dimensional.

BED LOAD EQUATION OF A A KALINSKE (1947)

A A Kalinske[2] (1947) developed an equation for computing the bed load sediment transport rate. He also presented a method for adapting the equation to sand mixtures. It is as follows:-

$$\frac{g_{btp} \cdot P}{v_* \gamma_s P_p D_p} = f\left(\frac{\tau_c}{\tau_o}\right) \qquad \ldots(11)$$

where g_{btp} = bed load transport rate in a range of grain sizes, dry weight per unit width per unit time.

P, P_p = factors which indicate the proportion of the bed taking the fluid shear.

D_p = effective particle size (diameter) within a given range of sizes.

The factor P was taken as 0.35 (Ref. 2) and P_p was evaluated using the expression

$$P_p = P \frac{p/D_p}{\Sigma(p/D_p)} \qquad \ldots(12)$$

Substituting equation (12) in (11) and using (6) to relate g_{btp} to the concentration by weight gives

$$X_b = \frac{v_*(s-1)}{Vd} \; \frac{\Sigma_1^n (p \cdot f(\tau_c/\tau_o)_p)}{\Sigma_1^n (p/D_p)} \qquad \ldots(13)$$

where n = number of fractions into which the grading curve is divided.

The value of τ_c was given by Kalinske as:-

$$\tau_c = \frac{1}{3}(\rho_s - \rho) g \, P_p \, D_p \qquad \ldots(14)$$

Hence, using (12) and substituting $\tau_o = \rho v_*^2$

$$\frac{\tau_c}{\tau_o} = \frac{1}{3} \cdot \frac{P(s-1)g}{v_*^2} \cdot \frac{p}{\Sigma_1^n (p/D_p)} \qquad \ldots(15)$$

for rough turbulent flow.

In programming for the computer, the graph expressing the function of τ_c/τ_o in the original paper of A A Kalinske was brought to the following analytical form:-

If $0.40 < \tau_c/\tau_o < 2.50$

$$f\left(\frac{\tau_c}{\tau_o}\right) = 10^{2.54(2.55-\tau_c/\tau_o)^{\frac{1}{2}} - 3.75} \qquad \ldots(16)$$

If $0 \leqslant \tau_c/\tau_o < 0.4$

$$f\left(\frac{\tau_c}{\tau_o}\right) = 10^{(0.375-\tau_c/\tau_o)/0.945} \qquad \ldots(17)$$

All the equations are dimensionally homogeneous so that any consistent set of units can be used and the calculation proceeds as follows:-

1. The representative grain size is chosen as the mean diameter of each size range into which the grading curve is divided.

2. The ratio τ_c/τ_o is evaluated using equation (15) with $P = 0.35$ for each fraction.

3. The function $f(\tau_c/\tau_o)$ is computed either with a graph (Ref 2) or by using equations (16) and (17).

4. The total concentration for the whole bed material load is computed from equation (13).

THE REGIME FORMULA OF C INGLIS (1947)

The regime theory rests on the principle of channel self-adjustment. According to E S Lindley[3] (1919) this principle can be expressed as follows:- "When an artificial channel is used to convey silty water, both bed and banks scour or fill; depth, gradient and width change until a state of balance is attained, at which the channel is said to be in regime".

In the discussion on the paper "Meanders and their bearing on river training" by C Inglis[4,5] 1947, as an answer to C M White, he suggested the following dimensionless set, in which the effects of the sediment "charge" was introduced into the Lacey regime equations of that time:

$$b = 2.67 \frac{Q^{\frac{1}{2}}}{g^{1/3}\ \nu^{1/12}} \left(\frac{X_s w}{D}\right)^{\frac{1}{4}} \qquad \ldots(18)$$

$$V = 0.7937 \frac{g^{7/18}\ Q^{1/6}}{\nu^{1/36}} (DX_s w)^{1/12} \qquad \ldots(19)$$

$$d = 0.4725 \frac{\nu^{1/9}\ Q^{1/3}\ D^{1/6}}{g^{1/18}\ (X_s w)^{1/3}} \qquad \ldots(20)$$

$$S_o = 0.000547 \frac{(DX_s w)^{5/12}}{\nu^{5/36}\ g^{1/18}\ Q^{1/6}} \qquad \ldots(21)$$

Algebraic manipulation of equations (18) and (19) produces:-

$$X_s = 0.562 \frac{\nu^{1/3}\ V^4}{w\ g^{5/3}\ d^2} \qquad \ldots(22)$$

in which w is the fall velocity of a characteristic sediment particle which is assumed to be the grain having the mean diameter (D_{50}) of the bed material.

The above formulae are for quartz sand in water, and can be used with any consistent set of units.

The calculation proceeds as follows:-

1. The fall velocity is computed either using Rubey's equations (A5) (See Appendix I) or from any experimental curve, for the mean diameter D_{50}.

2. The concentration is determined from equation (22).

BED LOAD EQUATION OF A MEYER-PETER AND R MULLER (1948)

Meyer-Peter and Muller[6] (1948) determined, for the bed load transport rate, an empirical relation which can be written as follows:

$$\gamma \frac{Q_s}{Q} \left(\frac{k_{se}}{k_r}\right)^{3/2} dS_o = 0.047 \, \gamma_s D + 0.25 \left(\frac{\gamma}{g}\right)^{1/3} \cdot q_{bt}^{2/3} \quad \ldots (23)$$

where q_{bt} = bed load transport rate, submerged weight per unit width.

Q_s = that part of Q whose energy is converted into eddying near the bed.

k_{se} = coefficient of total roughness due to skin and form roughness in the Strickler formula.

k_r = coefficient of particle friction with plane bed in the Strickler formula.

D = effective diameter of the sediment given by $D = \Sigma_1^p \, p \, . \, D_p$ in which p is the fraction of bed material in a size range D_p.

Now, $$q_{bt} = X_b V d \rho g \cdot \frac{(s-1)}{s} \qquad \ldots(24)$$

Hence using equations (4) and (24), equation (23) reduces to the form:-

$$X_b = 8\frac{s v_*^{\,3}}{(s-1)Vdg}\left\{\frac{Q_s}{Q}\left(\frac{k_{se}}{k_r}\right)^{3/2} - \frac{0.047}{v_*^{\,2}/gD(s-1)}\right\}^{3/2} \qquad \ldots(25)$$

According to the authors of Reference (6) the ratio $\frac{Q_s}{Q}$ is determined for rectangular channels as follows:-

$$\frac{Q_s}{Q} = \frac{b\,k_w^{3/2}}{2d\,k_{se}^{3/2} + b\,k_w^{3/2}} \qquad \ldots(26)$$

in which k_w is the side wall roughness

and k_{se} is computed using the following expressions:-

$$k_m = \frac{V}{R^{2/3}\,S_o^{\frac{1}{2}}} = \frac{Vg^{\frac{1}{2}}}{v_*\,R^{1/6}} \qquad \ldots(27)$$

$$k_{se} = \frac{k_m\,k_w\,b^{2/3}}{[bk_w^{3/2} + 2d(k_w^{3/2}-k_m^{3/2})]^{2/3}} \qquad \ldots(28)$$

When $b \to \infty$ equation (28) reduces to (27) and for practical purposes, natural rivers, k_{se} can be taken as k_m when $\frac{b}{d} \geqslant 15$. It can be proved that in this case the error in the computed value of k_{se} is about 10 per cent. For typical prototype sizes equation (26) also reduces to unity and $Q_s = Q$.

Most experimental flumes are constructed of concrete, glass or steel so that k_w can be estimated as:-

$$k_w = \frac{1}{n} = \frac{1}{0.01} = 100 \qquad \ldots(29)$$

where n is the Manning coefficient.

The value of k_r is given by

$$k_r = \left(\frac{8g}{\lambda}\right)^{\frac{1}{2}} \left(\frac{1}{R} \cdot \frac{Q}{Q_s}\right)^{1/6} \qquad \ldots(30)$$

The friction factor λ is obtained from the well-known equation of Nikuradse in which it is expressed as a function of the Reynolds number and the relative roughness

$$\frac{1}{\sqrt{\lambda}} = -2 \log\left(\frac{D_{90}\ Q}{14.8\ RQ_s} + \frac{0.6275\nu}{RV\sqrt{\lambda}}\right) \qquad \ldots(31)$$

In the region of fully developed turbulence k_r can be calculated with:-

$$k_r = \frac{26}{D_{90}^{1/6}} \text{ (metric units)} \qquad \ldots(32)$$

Except equations (27), (28), (29), (30) and (32) which are in m and m/s, all the equations are dimensionally homogeneous so that any consistent set of units may be used with them.

The calculation proceeds as follows:

1. The value of D, effective diameter of the sediment, is determined by $D = \Sigma_1^p\ p\ D_p$.

2. The values of k_{se} and the ratio Q_s/Q are determined from equations (28) (with k_w = 100) and (26) respectively, when the sediment transport rate in experimental flume is required. In natural rivers $k_{se} = k_m$ using equation (27) with $R = d$ and $Q_s = Q$.

3. The value of k_r is evaluated using equations (30), (31) or (32).

4. The concentration by weight is computed by using equation (25).

BED LOAD EQUATION OF H A EINSTEIN

The bed load relationship developed by H A Einstein is derived using considerations of probabilities, (Ref 7). His method, proposed initially in 1942, was considerably modified in his later works dating from 1950 onwards.

In this method the bed load discharge is computed for individual size fractions within the whole bed material. This means that the size distribution of bed load is also obtained.

The H A Einstein Bed-Load function is as follows:-

$$\frac{A_* \phi_*}{1 + A_* \phi_*} = 1 - \frac{1}{\sqrt{\pi}} \int_{-B_* \psi_* - \frac{1}{\eta_o}}^{B_* \psi_* - 1/\eta_o} e^{-t^2} dt \qquad \ldots(33)$$

where

$$\phi_* = \frac{p_B}{p_b} \frac{q_{bt} \rho^{\frac{1}{2}}}{(\gamma_s D_p)^{3/2}} \qquad \ldots(34)$$

$$\psi_* = \xi_p \; y \; \frac{\beta^2}{\beta_*^2} \frac{(s-1) D_p g}{(v_*')^2} \qquad \ldots(35)$$

where p_B = fraction of the bed load of a given grain size,

p_b = fraction of the bed material of a given grain size,

ϕ_* = intensity of transport for individual grain size,

v_*' = shear velocity with respect to the grain,

ξ = "Hiding factor" of grain in a mixture,

y = pressure correction in transition (smooth-rough).

According to Einstein,

$$\frac{\beta^2}{\beta_*^2} = \left[\frac{\log 10.6}{\log\left(10.6\ \frac{X}{\Delta}\right)}\right]^2 \qquad ...(36)$$

and $B_* = 0.143$, $A_* = 43.50$, $\eta_o = 0.50$.

Substituting from equation (24) the bed load concentration associated with a particular size fraction becomes:-

$$X_{bp} = p_B X = \frac{p_b \phi_* (s-1)^{\frac{1}{2}}\ s\ D_p^{3/2}\ g^{\frac{1}{2}}}{Vd} \qquad ...(37)$$

and the total concentration of bed load

$$X_b = \Sigma_1^p\ X_{bp} \qquad ...(38)$$

Einstein considered that the velocity distribution in an open channel is described by the logarithmic formulae based on V Karman's similarity theorem with the constant as proposed by Keulegan[8]. He gave the mean velocity including the transition between the rough and smooth boundaries as:

$$\frac{V}{v_*'} = 5.75 \log\left(12.27\ \frac{d'}{\Delta}\right)^{\dagger} \qquad ...(39)$$

in which:-

d' = mean depth with respect to the grain,

v_*' = shear velocity with respect to the grain,

Δ = the apparent roughness of the surface,

where $$\Delta = \frac{k_s}{x} \qquad ...(40)$$

k_s = the roughness of the bed.

x = function of k_s/δ'

δ' = the thickness of the laminar sublayer at a smooth wall, $11.6\ \nu/v_*'$. ...(41)

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† In the original works R' is used rather than d'.

The value of χ in equation (36) can be calculated according to Einstein as follows:-

$\chi = 0.77\Delta$ if $\Delta/\delta' > 1.80$

$\chi = 1.39\delta$ if $\Delta/\delta' < 1.80$ or if there is a uniform grain size ...(42)

and the values of the two correction factors in equation (35) are given by:-

$$\xi = \frac{f^{(i)}(D/\chi)}{f^{(ii)}(v_* D/\nu)} \qquad \text{...(43)}$$

$$y = f^{(iii)}(k_s/\delta') \qquad \text{...(44)}$$

H A Einstein presented graphs for the evaluation of x, ξ and y. The hiding factor, ξ, was introduced to define the influence of the mutual interference between the particles of different sizes with respect to the buoyancy forces. The original paper of Einstein(4) gave the factor ξ as a function of D/χ only. In his later works carried out with Ning Chien(9) in 1954 he introduced a second correction factor, this being a function of $v_* D/\nu$. Equation (43) represents the more recent concept.

The Einstein method includes a procedure for computing mean velocity, obtaining v_*'' from a bar resistance curve by trial and error and hence v_*' since v_* is known. The v_*' value enables x to be calculated and the mean velocity can then be determined. However, in the present exercise, measured mean velocities were available and the following procedure was adopted for determining v_*'.

$$(v_*')^2 = gd'S_o \qquad \text{...(45)}$$

$$v_*^2 = gdS_o \qquad \text{...(46)}$$

Hence $$d' = d \cdot \frac{(v_*')^2}{v_*^2} \qquad \text{...(47)}$$

Substituting equations (47) and (40) into (39) gives

$$\frac{V}{v'_*} = 5.75 \log\left(\frac{12.27d}{D_{65}} \cdot \frac{(v'_*)^2}{{v_*}^2} \cdot x\right) \qquad \ldots(48)$$

(k_s has been taken as D_{65}).

The value of x as a function of D_{65}/δ' is evaluated from either the graph given by Einstein or the following set of equations:-

If $D_{65}/\delta' < 0.4$; $x = 1.70 \log D_{65}/\delta' + 1.90$

If $0.4 \leq D_{65}/\delta' < 2.35$; $x = 1.615 - 1.544 (\log D_{65}/\delta')^{1.6}$

If $2.35 \leq D_{65}/\delta' < 10$; $x = 0.926(\log D_{65}/\delta'-1)^{2.43} + 1.00$

If $10 < D_{65}/\delta'$; $x = 1.00$. ...(49)

Vanoni and Brooks (Ref 23) give a direct graphical method and the reader interested in this approach should refer to this work. In programming for computer the graphs given by Einstein for determining ξ, y and $\theta = f^{(iii)}(v_*D/\nu)$ from equations (43) and (44) were expressed analytically as follows:-

1. Hiding factor, ξ

If $0.10 \leq \frac{D}{X} < 0.73$; $\xi = 0.70 \dfrac{\left(\frac{D}{X}\right)^{-2.385}}{\theta}$

If $0.73 \leq \frac{D}{X} < 1.30$; $\xi = 1.20 \dfrac{\left(\frac{D}{X}\right)^{-0.692}}{\theta}$

If $1.30 < \frac{D}{X}$; $\xi = \dfrac{1.00}{\theta}$ ...(50)

with:-

If $\frac{v_*D}{\nu} \leq 3.50$; $\theta = 10^{\left(\frac{0.544 - \log \frac{v_*D}{\nu}}{0.412}\right)}$

If $\frac{v_* D}{\nu}$ > 3.50 ; $\theta = 1.00$ ...(51)

2. Pressure correction in transition (smooth-rough)

If $\frac{D_{65}}{\delta'} \leqslant 0.47$; $y = \left(\frac{D_{65}}{\delta'}\right)^{1.187}$

If $0.47 < \frac{D_{65}}{\delta'} \leqslant 1.70$; $y = 10^{-2.23\left(\log \frac{D_{65}}{\delta'} - 0.0492\right)^2 - 0.083}$

If $1.70 < \frac{D_{65}}{\delta'} \leqslant 3.15$; $y = 0.8\left(\frac{D_{65}}{\delta'}\right)^{-0.378}$

If $3.15 < \frac{D_{65}}{\delta'} \leqslant 5.00$; $y = 0.525$

If uniform grain size; $y = 1.00$ ...(52)

All the equations are dimensionally homogeneous so that any consistent set of units may be used. The value of the integral in equation (33) was evaluated by expanding the integrand in Maclaurin's series and integrating term by term (see Appendix II).

The calculation proceeds as follows:-

1. From the granulometric curve D_{65} is taken as the skin roughness k_s.

2. The friction velocity v_*' is calculated either using the method described previously or the direct graphical method given in Ref (23).

3. The thickness of the laminar sublayer δ is obtained from equation (41).

4. The correction factor x is evaluated either from the graph given by Einstein or from the equation set (49).

5. The apparent roughness Δ is obtained using equation (40).

6. From the ratio Δ/δ' the value of χ is calculated using the set equation (42).

7. The pressure correction factor is determined using the set of equations (52).

8. The ratio β^2/β_*^2 is calculated from equation (36).

9. The representative grain sizes are chosen as the mean diameter of each size fraction.

10. The "hiding factor" ξ of each fraction is determined from the sets of equations (50) and (51).

11. For each grain size fraction the parameter ψ_* is calculated using equation (35).

12. The intensity of transport, ϕ_*, for individual grain size is evaluated from equation (31) (see Appendix II).

13. The concentration by weight for each fraction and the total concentration for the whole bed material are then computed using equations (37) and (38).

TOTAL LOAD FORMULA OF H A EINSTEIN (1950)

In his 1950 paper (Ref 7) Einstein also gave a method of determining the total sediment load, excluding wash load. The total load in a particular grain size range was given as:-

$$p_T\, g_{st} = p_B\, g_{bt}\{PI_1 + I_2 + 1\} \qquad \ldots(53)$$

in which p_T = fraction of the total load in a given grain size

g_{st} = total load, dry weight per unit width per unit time.

$$I_1 = 0.216 \frac{A^{Z-1}}{(1-A)^Z} \int_A^1 \left(\frac{1-t}{t}\right)^Z dt \qquad \ldots(54)$$

$$I_2 = 0.216 \frac{A^{Z-1}}{(1-A)^Z} \int_A^1 \left(\frac{1-t}{t}\right)^Z \ln t \, dt \qquad \ldots(55)$$

where A = ratio of bed layer thickness to water depth

$$A = 2D/d \qquad \ldots(56)$$

$$Z = 2.5w/v_*' \qquad \ldots(57)$$

$$P = \frac{1}{0.434} \log\left(\frac{30.2}{\Delta/d}\right) \qquad \ldots(58)$$

The transport rate is related to the concentration by weight using equation (24) and making these substitutions in equation (53) we obtain

$$X_{sp} = X_{bp}\{PI_1 + I_2 + 1\} \qquad \ldots(59)$$

and the total bed material load for all the fractions is given by

$$X_s = \Sigma_1^p X_{sp} \qquad \ldots(60)$$

The two integrals in equations (54) and (55) were solved by numerical integration using Simpson's Rule (see Appendix II).

The calculation proceeds as follows:-

1. The settling velocity is calculated from equation (A5) using the mean diameter for each fraction of the bed sample (see Appendix I).

2. The values of Z and A are calculated using equations (57) and (56) for each fraction.

3. The integrals I_1 and I_2, equations (54) and (55) are evaluated by numerical integration (see Appendix II) or by the original graph of H A Einstein (Ref 7).

4. The value of P is determined from equation (58).

5. Introducing the values of X_{bp} from equation (37) in equation (59) the concentration, X_{sp}, is obtained, and from equation (60) the concentration for all the fractions.

BED LOAD EQUATION OF H A EINSTEIN AND C B BROWN (1950)

The bed load equation of Einstein-Brown is a modification developed by H Rouse, M C Boyer and E M Laursen of the formula of H A Einstein. It was presented by C B Brown[11] in 1950 and appears to be based on empirical considerations. The equations can be written as follows:-

$$\frac{\phi}{F} = f\left(\frac{1}{\psi}\right) \qquad \ldots(61)$$

where

$$\phi = \frac{g_{bt}}{g^{3/2}\ \rho_s (s-1)^{\frac{1}{2}}\ D_{50}^{3/2}} \qquad \ldots(62)$$

$$1/\psi = \frac{v_*^{\ 2}}{(s-1) D_{50}\ g} \qquad \ldots(63)$$

$$F = \left(\frac{e}{3} + \frac{36\ \nu^2}{g\ D_{50}^3 (s-1)}\right)^{\frac{1}{2}} - \left(\frac{36\ \nu^2}{g\ D_{50}^3 (s-1)}\right)^{\frac{1}{2}} \qquad \ldots(64)$$

The relationship between ϕ, F and ψ was given by the authors of Reference 11 in a semi-graphical form. When $1/\psi$ exceeds 0.1 it becomes $\phi/F = 40(1/\psi)^3$. To programme the relationship when $1/\psi$ is less than 0.1 the function was approximated by an exponential equation

$$\frac{\phi}{F} = 10^{-4.0 + (23.695 + 16.925 \log \frac{1}{\psi})^{\frac{1}{2}}} \qquad \ldots(65)$$

Combining the above equations and relating the bed load transport to the concentration by weight we obtain the following predictive equations:-

$$\text{If } \frac{v_*^{\,2}}{gD_{50}(s-1)} \geq 0.1 \; ; \; X_{bt} = \frac{40\, v_*^{\,6}\, F\, s}{g^{5/2} Vd\,(s-1)^{5/2} D_{50}^{3/2}}$$

$$\text{If } \frac{v_*^{\,2}}{gD_{50}(s-1)} < 0.1 \; ; \; X_{bt} = \frac{g^{\frac{1}{2}} D_{50}^{3/2} (s-1)^{\frac{1}{2}} s}{Vd} \cdot \phi \qquad \ldots(66)$$

where ϕ is determined from equation (65)

For the above expressions the quantity F appears in the Rubey equation (see Appendix I) and is defined as a dimensionless function of fall velocity.

The calculation proceeds as follows:-

1. The dimensionless function of the fall velocity is calculated using equation (64).

2. The concentration of bed load transport by weight is evaluated using the equations (62) to (66) inclusive.

TOTAL LOAD FORMULA OF A A BISHOP, D B SIMONS AND E V RICHARDSON (1965)

The quantities A_* and B_* which appear in the relation (33) were assumed by H A Einstein as constants. Later works of A A Bishop, D B Simons and E V Richardson (Ref 10) revealed that the relation (33) can be improved if A_* and B_* are treated as variable quantities.

The authors of Reference (10) reasoning that "the instantaneous variations in the lift forces may lift some particles from the bed into suspension whereas other particles will be moved only within the bed layer", concluded that the relationship $\phi_* - \psi_*$ should be concerned with the total bed material sediment transport rate rather than with the bed load transport rate. Working on this basis they found a relation between A_* and B_* and the grain size D.

The experiments reported do not make clear the influence of such parameters as specific gravity and temperature. The graph of this relation is expressed here in the following analytical form:

$$\text{If } D_{50} < 1 \text{ mm} \quad ; \; B_* = \frac{0.0375}{D_{50\,(mm)}} + 0.1054$$

$$\text{If } D_{50} \geqslant 1 \text{ mm} \quad ; \; B_* = 0.143$$

$$\text{If } D_{50} < 1.12 \text{ mm} \; ; \; A_* = \left(\frac{D_{50\,(mm)}}{0.2} + 1\right)^2$$

$$\text{If } D_{50} \geqslant 1.12 \text{ mm} \; ; \; A_* = 43.5 \qquad \ldots(67)$$

The method outlined by Bishop, Simons and Richardson involves:-

(i) The computation of ψ' as in the Einstein method but, in order to utilise the measured mean velocities available from experiments, v_*' was computed by the method presented by Vanoni and Brooks (Ref 23). ψ' is then given by

$$\psi' = D_{35}(s-1)g/(v_*')^2 \qquad \ldots(68)$$

(ii) The use of expression (33) in the following form:-

$$\frac{A_*\phi}{1 + A_*\phi} = 1 - \frac{1}{\sqrt{\pi}} \int_{-B_*\psi' - 1/\eta_o}^{B_*\psi' - 1/\eta_o} e^{-t^2} dt \qquad \ldots(69)$$

in which ϕ is given by

$$\phi = \frac{q_{st}\rho^{\frac{1}{2}}}{(\gamma_s D_{50})^{3/2}} = \frac{q_{bt}}{g^{3/2}\rho_s (s-1)^{\frac{1}{2}} D_{50}^{\frac{1}{2}}} \qquad \ldots (70)$$

and A_* and B_* are given by the set of equations (67). Thus the values of p_B and p_b are assumed equal to unity and the concentration of the total load is given by equation (37) in which $p_b = 1.0$.

According to the authors of Reference (10) the relationship described by equation (69) is valid only when ripples and dunes exist. In the transition regime and for later stages of motion systematic errors develop and the relationships have been modified to fit the data empirically.

In the original works the relationships between ϕ, ψ' and D_{50} are given graphically.

The calculation proceeds as follows:-

1. The value of v_*' is computed using either equations (45) to (48) or the graphs presented in Reference (23).

2. The shear intensity factor ψ' is determined using equation (68).

3. The values of A_* and B_* are obtained from the set of equations (67).

4. The intensity of transport for the total bed material load is evaluated using either equations (69) and (70) or the graphs given by Bishop, Simons and Richardson in Reference (10).

5. The concentration is computed from equation (37) putting $p_b = 1$ and $D_p = D_{50}$.

BED LOAD EQUATION OF R A BAGNOLD (1956)

R A Bagnold[12] (1956) gave a formula to express the bed load transport rate. It was based on principles of energy and may be written as follows:-

$$\frac{q_{bt}\ \rho^{\frac{1}{2}}}{(\gamma_s\ D)^{3/2}} = \alpha Y^{\frac{1}{2}}(Y-Y_C) \qquad \ldots(71)$$

in which Y = Shield's mobility parameter given by $Y = v_*^{\ 2}/(gD(s-1))$

Y_C = The value of Y at initial motion

q_{bt} = bed load transport rate, submerged weight per unit width

$$\alpha = 8.5 e_b/t_g\psi_o \qquad \ldots(72)$$

e_b = bed load transport efficiency

$t_g\psi_o$ = coefficient of dynamic friction.

Substituting the Shield's parameter into equation (71) and noting that the bed load transport rate is related to the concentration by weight through

$$q_{bt} = X_b V d \rho g \frac{(s-1)}{s} \qquad \ldots(73)$$

we obtain

$$X_b = \frac{\alpha\ v_*^{\ 3}\ s}{(s-1)Vdg}\left\{1 - \frac{Y_C}{v_*^{\ 2}/(gD(s-1))}\right\} \qquad \ldots(74)$$

where D is taken as $D = \Sigma D_p\ p$ in which p is the proportion by weight of particles in a size range D_p.

R A Bagnold related $e_b/t_g\psi_o$ to particle size and presented the functional relationships graphically. In programming for computer these relationships were introduced into equation (72) in the following form:-

If $D < 0.5$ mm ; $\alpha = 8.50\, D$ (mm)

If $D \geq 0.5$ mm ; $\alpha = 4.25$ ...(75)

This theory has the following restrictions:-

(i) The ratio $e_b/t_g\psi_o$ is given for sand and water. It is thus only applicable where $\gamma_s = 1.65$.

(ii) It is claimed to be valid only for the earlier stages of the movement, say $Y \leq 0.4$.

(iii) In his derivation R A Bagnold assumed a rough plane bed (without sand waves) and rough turbulent conditions.

The calculation proceeds as follows:-

1. The value of the effective diameter of the sediment forming the bed of the channel is determined from $D = \Sigma_1^p pD_p$.

2. The value of α is determined from the set of equations (75).

3. The dimensionless critical tractive force Y_c is calculated either from the equation A4 (see Appendix I) or from the original graphs presented by Shields (Ref 1).

4. The concentration by weight is evaluated using equation (74).

TOTAL LOAD FORMULA OF R A BAGNOLD (1966)

R A Bagnold[13] has developed a theory to express the total load of bed material which is based on similar principles to the bed load equation.

The total load equation can be written as:-

$$\frac{q_{st}}{\omega} = \frac{e_b}{t_g\psi_o} + e_s(1 - e_b)\,\frac{V}{w} \qquad \ldots(76)$$

where q_{st} = total load (bed material), submerged weight per unit width per unit time

ω = stream power per unit boundary area

e_b = bed load transport efficiency

e_s = suspended load transport efficiency

$t_g\psi_o$ = coefficient of dynamic friction

w = effective fall velocity

The stream power, ω, is related to the hydraulic properties as follows:-

$$\omega = \frac{\rho g Q S_o}{b} = \rho g d V S_o \qquad \ldots(77)$$

and substituting $\qquad v_*^2 = gdS_o$

$$\omega = \rho V v_*^2 \qquad \ldots(78)$$

The transport rate for the total load (bed material) is related to the concentration X_s by

$$q_{st} = X_s V d \rho g \frac{(s-1)}{s} \qquad \ldots(79)$$

Substituting (78) and (79) in (76)

$$X_s = \frac{v_*^2\, s}{gd(s-1)} \left\{ \frac{e_b}{t_g\psi_o} + e_s(1-e_b)\frac{V}{w} \right\} \qquad \ldots(80)$$

In his later works R A Bagnold relates $t_g\psi_o$ to a parameter closely analogous to the fluid Reynolds number

$$t_g\psi_o = f\{G^2\} \qquad \ldots(81)$$

where

$$G^2 = \frac{s\, D^2\, v_*^2}{14\, \nu^2} \qquad \ldots(82)$$

R A Bagnold gave graphs to determine e_b and $t_g\psi_o$ but these have been brought to analytical form as follows:-

(i) $t_g\psi_o$

If $G^2 < 150$; $t_g\psi_o = 0.75$

If $150 \leqslant G^2 < 6000$; $t_g\psi_o = -0.236 \log G^2 + 1.250$

If $G^2 > 6000$; $t_g\psi_o = 0.374$ ...(83)

The theory is only claimed to be applicable at high transport rates, the applicable range being defined by the expression

$$G^2 > \frac{0.65}{14} \cdot \frac{s(s-1)D^3 g}{\nu^2} \cdot t_g\psi_o \qquad \ldots(84)$$

(ii) e_b (Only for $\gamma_s = 1.65$ and $0.3 \leqslant V(m/s) \leqslant 3.0$)

If $D < 0.06\,(mm)$; $e_b = -0.012 \log 3.28V + 0.150$

If $0.06 \leqslant D(mm) < 0.2$; $e_b = -0.013 \log 3.28V + 0.145$

If $0.2 \leqslant D(mm) < 0.7$; $e_b = -0.016 \log 3.28V + 0.139$

If $0.7 < D(mm)$; $e_b = -0.028 \log 3.28V + 0.135$

where V = mean velocity of flow (m/s)

D = effective particle diameter.

In order to bring his transport formula into a form that can be used for practical purposes, Bagnold assumed, ignoring certain uncertainties, that the suspension efficiency e_s has the universal constant value 0.015 for fully developed suspension by turbulent shear flow, and took for the numerical coefficient in the second term of equation (76) the round figure of $\frac{2}{3}$ 0.015 = 0.01.

Equation (76) thus becomes:-

$$X_s = \frac{v_*^2 \, s}{gd(s-1)} \left\{ \frac{e_b}{t_g \psi_o} + 0.01 \, \frac{V}{w} \right\} \qquad \ldots(86)$$

The quantities D and w are taken as follows:-

$$D = \frac{\Sigma_1^p \, pD_p}{\Sigma_1^p \, p} \text{ of the whole bed material} \qquad \ldots(87)$$

$$w = \frac{\Sigma_1^p \, p \, w_p}{\Sigma_1^p \, p} \text{ of the suspended material} \qquad \ldots(88)$$

Or

$$w = \tfrac{1}{2} \, \frac{\Sigma_1^p \, p \, w_p}{\Sigma_1^p \, p} \text{ of the whole bed material} \qquad \ldots(89)$$

All equations are dimensionless except the set (85) where the units are m/s. The total load theory has the following restrictions:-

(i) It is not claimed to be valid at the early stages of movement, say $v_*^2/gD(s-1)) < 0.4$,

(ii) there is no evidence to support the theory at depths below 150 mm,

(iii) e_b values are related to sand transport in water, $\gamma_s = 1.65$.

The computation proceeds as follows:-

1. The effective grain diameter and fall velocities are computed from the grading curve of the bed material using equations (87), (88) and (89) and A5 (see Appendix I).

2. The bed load transport efficiency is evaluated either from the original graph of R A Bagnold (Ref 13) or by using the set of equations (85).

3. The value of G^2 is determined from equation (82).

4. The coefficient of dynamic friction $t_g\psi_o$ is evaluated using equations (83) and (84) or the graphs in Ref (13).

5. Concentrations are determined from equation (86).

THE TOTAL LOAD FORMULA OF E M LAURSEN (1958)

The method of calculating total load proposed by E M Laursen (Ref 14) rests on empirical relationships. It predicts both the quantity and composition of the sediment in transport.

The basic equation can be written:-

$$X_{st} = 0.01 \sum_{1}^{p} p \left(\frac{D_p}{d}\right)^{7/6} \left(\frac{\tau_o'}{\tau_c'} - 1\right) \cdot f\left(\frac{V_*}{w_p}\right) \qquad \ldots(90)$$

where τ_o' = tractive shear associated with the sediment grains,

τ_c' = critical shear associated with the grains.

$$\tau_o' = \frac{\rho V^2}{58}\left(\frac{D_{50}}{d}\right)^{1/3} \qquad \ldots(91)$$

$$\tau_c' = Y_C \rho g (s-1) D_p \qquad \ldots(92)$$

Substituting from equations (91) and (92) into (90) yields:-

$$X_{st} = 0.01 \sum_{1}^{p} p \left(\frac{D_p}{d}\right)^{7/6} \left(\frac{V^2}{58 Y_C D_p g (s-1)}\left(\frac{D_{50}}{d}\right)^{1/3} - 1\right) \cdot f\left(\frac{V_*}{w_p}\right) \qquad \ldots(93)$$

The value of Y_C was related by Laursen to D_p/δ as follows:-

$$\begin{aligned}
&\text{If } D_p/\delta > 0.1 &&; Y_C = 0.04 \\
&\text{If } 0.1 \geq D_p/\delta > 0.03 &&; Y_C = 0.08 \\
&\text{If } 0.03 > D_p/\delta &&; Y_C = 0.16
\end{aligned} \qquad \ldots(94)$$

where δ = thickness of the laminar sublayer

$$\delta = 11.6\ \nu/v_* \qquad \ldots(95)$$

Laursen gave the function of v_*/w_p in equation (93) in a graphical form, the analytical equivalent of which can be written:-

$$\text{If } 10^{-2} \leq v_*/w_p \leq 0.3; f\left(\frac{v_*}{w_p}\right) = 10.8\left(\frac{v_*}{w_p}\right)^{0.253}$$

$$\text{If } 0.3 \leq v_*/w_p < 3.0; f\left(\frac{v_*}{w_p}\right) = 10^{0.97 \log\frac{v_*}{w_p} + 0.85 \log^2\frac{v_*}{w_p} + 1.20}$$

$$\text{If } 3.0 \leq v_*/w_p < 20; f\left(\frac{v_*}{w_p}\right) = 5.6\left(\frac{v_*}{w_p}\right)^{2.30}$$

$$\text{If } 20 \leq v_*/w_p < 10^3; f\left(\frac{v_*}{w_p}\right) = 10^{3.16 \log\frac{v_*}{w_p} - 0.57 \log^2\frac{v_*}{w_p} + 0.413}$$

$$\ldots(96)$$

The ρ in equation (91) was not in the original equation suggested by Laursen. It has been introduced in this report so that the equation becomes dimensionally homogeneous and Laursen's coefficient has been changed accordingly. The Laursen method, as it stands at present, is only applicable to natural sediments with a specific gravity of 2.65.

The calculation proceeds as follows:-

1. Representative grain sizes D_p are chosen as the mean size of each size fraction, p.

2. The thickness of the laminar sublayer is computed using equation (95).

3. The values of Y_C for each fraction, p, are determined using the set of equations (94).

4. The values of $f(v_*/w_p)$ are determined for each size fraction using the set of equations (96), the fall velocity being determined using equation A5 (see Appendix I).

5. The total concentration is evaluated using equation (93). This represents the sum of the concentrations of the individual size fractions.

The first equation in set (96)

$$f\left(\frac{v_*}{w_p}\right) = 10.8\left(\frac{v_*}{w_p}\right)^{0.253}$$

gives the bed load transport rate, according to Laursen, so long as conditions are within the range

$$10^{-2} \leq v_*/w_p < 10^3$$

This seems to indicate that if $v_*/w_p < 0.3$ the transport of material is solely a bed process.

THE BED LOAD EQUATION OF J ROTTNER (1959)

J Rottner (Ref 15) has developed an equation to express the bed load transport rate in terms of the fluid and sediment properties which is based on dimensional considerations with certain empirical coefficients introduced to satisfy available data. The main interest in Rottner's work is that he carried out a systematic investigation into the effect of the geometrical ratio d/D.

The expression given by Rottner is:-

$$\frac{M}{\rho_s \sqrt{gd^3 (s-1)}} = \left\{ \left[0.667 \left(\frac{D_{50}}{d} \right)^{2/3} + 0.14 \right] \frac{V}{(s-1)^{\frac{1}{2}} (gd)^{\frac{1}{2}}} - 0.778 \left(\frac{D_{50}}{d} \right)^{2/3} \right\}^3 \qquad \ldots(97)$$

where M = sediment transport rate, mass per unit width per unit time,

D_{50} = mean grain size diameter.

Substituting $\quad M = XVd\rho \qquad \ldots(98)$

and rearranging equation (97) yields:-

$$X = \frac{s(s-1)^{\frac{1}{2}} (gd)^{\frac{1}{2}}}{V} \left\{ \left[0.667 \left(\frac{D_{50}}{d} \right)^{2/3} + 0.14 \right] \frac{V}{(gd(s-1))^{\frac{1}{2}}} - 0.778 \left(\frac{D_{50}}{d} \right)^{2/3} \right\}^3 \qquad \ldots(99)$$

The above equation is dimensionally homogeneous and any consistent set of units can be used. In his derivation, wall effects were ignored and according to Rottner the theory "may not be applicable to uses where only small quantities of material are being conveyed".

The calculation of concentrations is achieved by direct substitution in equation (99).

THE BED LOAD FORMULA OF M S YALIN (1963)

M S Yalin developed a bed load equation based on a theoretical analysis of saltating particles. The final expression is:-

$$\frac{q_{bt}}{\gamma_s D v_*} = 0.635\ s^* \left\{1 - \frac{\ln(1 + as^*)}{as^*}\right\} \qquad \ldots(100)$$

where
$$s^* = \frac{v_*^2/(gD(s-1))}{Y_C} - 1 \qquad \ldots(101)$$

$$a = 2.45\ \frac{Y_C^{\frac{1}{2}}}{s^{0.4}} \qquad \ldots(102)$$

Relating the bed load transport rate to the concentration using equation (24), equation (100) becomes

$$X_b = 0.635\ \frac{D v_*\ s\ s^*}{Vd} \left\{1 - \frac{\ln(1 + as^*)}{as^*}\right\} \qquad \ldots(103)$$

This equation is dimensionally homogeneous. The formula is restricted (i) to plane bed conditions, (ii) to fully developed turbulent flow and (iii) large depth/diameter ratios.

Equation (103) was developed to apply to material of uniform grain size. When the method is applied to a graded sediment Yalin suggested that "an effective or typical" grain size should be used. He was not specific on this point and the mean diameter (D_{50}) has been used in the present analysis.

The calculation proceeds as follows:-

1. The mean diameter (D_{50}) is determined from the grading curve of the bed material.

2. Shields critical Mobility number (Y_C) is calculated using the set of equations (A4) or from Shields graphical representation (Ref 1).

3. The values of s* and a are computed from equations (101) and (102).

4. Equation (103) gives the bed load concentration.

THE REGIME FORMULA OF T BLENCH (1964)

Based on the data and regime principles described by C Inglis and G Lacey, T Blench (Ref 17) gave the following three basic equations:-

$$V^2/d = F_b \qquad \ldots(104)$$

$$V^3/b = F_s \qquad \ldots(105)$$

$$\frac{V^2}{gdS_o} = \frac{3.63}{K}\left(1 + \frac{10^5 X}{233}\right)\left(\frac{Vb}{\nu}\right)^{\frac{1}{4}} \qquad \ldots(106)$$

where F_b = bed factor

F_s = side factor

b = mean breadth

d = mean depth

K = meander slope correction.

Algebraic manipulation of equations (104) and (106) yields:-

$$\frac{F_b^{\,11/12}}{1 + \frac{10^5 X}{233}} = \frac{3.63\ g\ b^{\frac{1}{4}}\ q^{1/12}\ S_o}{K\nu^{\frac{1}{4}}} \qquad \ldots(107)$$

Substituting $q = Vd$ and $v_* = (gdS_o)^{\frac{1}{2}}$ then gives

$$\frac{F_b^{\,11/12}}{1 + \frac{10^5 X}{233}} = \frac{3.63\ b^{\frac{1}{4}}\ V^{1/12}\ v_*^{\,2}}{K\ \nu^{\frac{1}{4}}\ d^{11/12}} \qquad \ldots(108)$$

Blench suggested the following values for K

K = 1.25 for a straight reach

K = 2.00 for a reach with well developed meandering without braiding

K = 3.00 for a braided reach

K = 4.00 for an extremely braided reach

The bed factor (F_b) is related to the "zero bed factor" (F_{bo}) by the expression:-

$$F_b = F_{bo}(1 + 0.12X10^5) \qquad ...(109)$$

for values of X less than about 10^{-4}. The "zero bed factor" is evaluated as follows:-

$$\text{If } D \leq 2(\text{mm}) \; ; \; F_{bo} = 1.9 \sqrt{D(\text{mm})} \qquad ...(110)$$

$$\text{If } D > 2(\text{mm}) \; ; \; F_{bo} = 0.58\, w_{70}^{11/24} (\nu_{70}/\nu)^{11/72} \qquad ...(111)$$

$$\text{or} \quad F_{bo} = 7.3\, D^{\frac{1}{4}} (\nu_{70}/\nu)^{1/6} \qquad ...(112)$$

These apply for:-

$$s = 2.65 \text{ and } F_{bo} \geq 38 \left(\frac{D}{d}\right)^{\frac{1}{2}} \qquad ...(113)$$

If the calculated value of F_{bo} using equations (111) or (112) does not satisfy equation (113) then the value of F_{bo} is taken as $38\left(\frac{D}{d}\right)^{1/12}$.

The units are confusing. In equation (110) D is in mm, in equation (111) w is in cm/s and in equation (112) D is in ft. The answer for F_{bo} is always in ft/s! w_{70} is the fall velocity of the median sand size (D_{50}) in water at 70 deg F and ν_{70} is the kinematic viscosity at this temperature. The limits of applicability of equations (110 to (112) are not given by Blench in a precise way.

Substituting (109) in (108)

$$\frac{(1 + 0.12X10^5)^{11/12}}{1 + \frac{10^5 X}{233}} = \frac{3.63\, v_*^2\, b^{\frac{1}{4}}\, V^{1/12}}{K\, d^{11/12}\, \nu^{\frac{1}{4}}\, F_{bo}^{11/12}} \qquad ...(114)$$

According to Blench the left hand side of equation (114) can be approximated as follows:-

If $10^{-5} < X \leqslant 10^{-4}$; LHS = $19.91\ X^{\frac{1}{4}}$ ...(115)

If $10^{-4} < X \leqslant 0.002$; LHS = $293\ 537\ X^{13/24}$ ...(116)

Blench interpreted X as the concentration of "that part of the total which is not suspended".

The equations have the following restrictions:-

(i) Equation (106) was established for low flow conditions.

(ii) Equation (109) is valid only for sand, not for coarse material. The value of the constant 0.12 in the same equation applies to subcritical flow conditions and concentrations less than 10^{-4}.

(iii) All the equations are unlikely to apply if b/d falls below about 4 or depth below about 0.4 m.

Except for F_{bo} the equations are dimensionally homogeneous. F_{bo}, equations (110) to (113), is in ft/s^2.

The calculation proceeds as follows:-

1. The bed factor F_b is calculated using equations (110) to (113).

2. The concentration X is determined by trial and error using equations (114) to (116).

TOTAL LOAD FORMULA OF F ENGELUND AND E HANSEN (1967)

F Engelund and E Hansen (Ref 18) developed an equation to express the total load (bed material) as follows:-

$$\frac{\lambda}{4}\ \phi = 0.1\ Y^{5/2} \qquad \text{...(117)}$$

$$\text{where } \lambda = \frac{8gdS_o}{V^2} \qquad \text{...(118)}$$

$$\phi = \frac{g_{st}}{\rho_s (s-1)^{\frac{1}{2}} g^{3/2} D_{50}^{3/2}} \qquad \ldots(119)$$

$$Y = \frac{v_*^{\,2}}{g\, D_{50}(s-1)} \qquad \ldots(120)$$

$$v_* = \sqrt{gdSo} \qquad \ldots(121)$$

Substituting from (118), (119), (120) and (121) into (117) yields:-

$$g_{st} = \frac{0.05\, \rho_s\, g^{\frac{1}{2}}\, V^2\, D_{50}^{\frac{1}{2}}\, Y^{3/2}}{(s-1)^{\frac{1}{2}}} \qquad \ldots(122)$$

But, as before,

$$g_{st} = X\, V\, d\, \rho\, g \qquad \ldots(123)$$

Hence,

$$X = \frac{0.05\, s\, V\, v_*^{\,3}}{dg^2\, D_{50}(s-1)^2} \qquad \ldots(124)$$

In their original paper Engelund and Hansen gave a graphical solution which facilitated the determination of both the water flow parameters and the sediment transport rates. In the present exercise mean velocities were available and were not computed by the Engelund and Hansen method.

Equation (124) is dimensionally homogeneous and the sediment transport concentration can be calculated directly from this equation using any consistent set of units.

THE TOTAL LOAD FORMULA OF W H GRAF (1968)

W H Graf (Ref 20) proposed a relationship to express the total transport rate (bed material) in both open and closed conduits as follows:-

$$\phi = 10.39(\psi)^{-2.52} \qquad \ldots(125)$$

or

$$\frac{g_{st}}{g^{3/2}\rho_s(s-1)^{\frac{1}{2}}D^{3/2}} = 10.39\left\{\frac{(s-1)gD}{v_*^2}\right\}^{-2.52} \qquad \ldots(126)$$

where ϕ and ψ have the same meaning as in the Einstein-Brown equations.

Using equation (123) and re-arranging, equation (126) becomes

$$X = \frac{10.39\ s(s-1)^{-2.02}g^{-2.02}D^{-1.02}}{Vd\ v_*^{-5.04}} \qquad \ldots(127)$$

W H Graf was not specific about the interpretation of the sediment diameter, D. However, he used total load data to derive his formula taking the mean diameter as the relevant size. Hence, in this report D has been assumed equal to the D_{50} size of the bed material.

The calculation proceeds directly from equation (127) using any consistent set of units.

THE TOTAL LOAD FORMULA OF F TOFFALETI (1968)

The total load formula (bed material) presented by F Toffaleti (Refs 21 and 22) is based on the concepts of H A Einstein (Ref 7) with certain, mainly empirical, modifications. There are three main differences from the Einstein method.

(i) The velocity profile in the vertical is represented by the relationship:-

$$v = (1 + Z_v)\, V \left(\frac{y_*}{d}\right)^{Z_v} \qquad \ldots(128)$$

where where $Z_v = 0.1198 + 0.00048\, T_F$ ...(129)

T_F = water temperature in degrees fahrenheit

v = flow velocity at a distance y_* above the bed.

(ii) The three Einstein correction factors β^2/β_*^2, ξ/θ and y are reduced to two viz:-

$$A_1 = f_1 \left\{ \left(\frac{10^5 g\nu}{32.2}\right)^{1/3} \Big/ 10 v_*' \right\} \qquad \ldots(130)$$

$$k = f_2 \left\{ \frac{\left(\frac{10^5 g\nu}{32.2}\right)^{1/3} v_*^2\, D_{65}\, 10^5}{10\, v_*'\, d\, g} \right\} \qquad \ldots(131)$$

in which D_{65} must be expressed in feet.

(iii) The relationship ϕ_* versus ψ_* was assumed between the level of $y_* = 2D$ and $y_* = d/11.24$ rather than between $y_* = 0$ and $y_* = 2D$ as in the Einstein theory.

The levels y_* equals 2D, d/11.24 and d/2.5 are used by Toffaleti as discontinuities within the sand concentration distribution. In each of the three zones the sand concentration is defined by the expression

$$C_{xp} = C_x \left(\frac{y_*}{d}\right)^{-aZ_p} \qquad \ldots(132)$$

where C_x and a are constants for each zone.

$$Z_p = \frac{w_p V}{C_z v_*^2} \qquad \ldots(133)$$

$$C_Z = \frac{1}{32.2} (260.67 - 0.667\ T_F) = 8.095 - 0.0207\ T_F \quad ...(134)$$

If Z_p (equation (133)) < Z_v (equation (129)), Z_p must be arbitrarily taken as:-

$$Z_p = 1.5\ Z_v \quad ...(135)$$

F Toffaleti gives his ϕ_* versus ψ_* relationship as:-

$$\phi_* = 17.17/\psi_*^{5/3} \quad ...(136)$$

where $$\psi_* = \frac{T\ A_1\ k\ D\ 10^4}{V^2} \quad ...(137)$$

and $$T = g(s-1)\ \overline{CL_2}\ (0.00158 + 0.0000028\ T_F) \quad ...(138)$$

Equation (136) can be re-written making substitutions from equations (137) and (34) as follows:-

$$GF_p = \frac{3.68\ \rho\ g^{3/2}\ s(s-1)^{\frac{1}{2}}\ D_*^{3/2}\ p_b\ 10^{-6}}{\left[\frac{TA_1 kD}{V^2}\right]^{5/3}} \quad ...(139)$$

where GF_p = sand fraction load within the depth range $y_* = 2D$ and $y_* = d/11.24$ for a particular grain size range, dry weight per unit width per unit time.

D_* = mean size of Sieve No 80 U.S. Standard (0.177 mm $\equiv$ 0.00058 ft).

For natural sand rivers s = 2.65 and in Imperial Units (tons/day/ft width) equation (139) reduces to:-

$$GF_p = \frac{0.600\ p_b}{\left(\frac{TA_1 kD}{0.00058V^2}\right)^{5/3}} \quad \text{(tons/day/ft width)} \quad ...(140)$$

which is the original relationship proposed by F Toffaleti. However, equations (136) and (139), are the more fundamental non-dimensional relationships. It does not follow, though, that the theory is necessarily applicable to all types of sediment since the functions A_1 and k were determined for sand only.

Using the above concepts, the sediment transport rates were calculated by Toffaleti for each depth zone as follows:-

(i) LOWER ZONE ($2D < y_* \leqslant d/11.24$)

$$g_{SLp} = p_b\, C_{Lp}(1+Z_v)Vd^{0.756Z_p - Z_v} \cdot N_L \qquad \text{...(141)}$$

where

$$N_L = \frac{\left(\frac{d}{11.24}\right)^{1+Z_v-0.756Z_p} - (2D_p)^{1+Z_v-0.756Z_p}}{1 + Z_v - 0.756\, Z_p} \qquad \text{...(142)}$$

(ii) MIDDLE ZONE ($d/11.24 < y_* \leqslant d/2.5$)

$$g_{SMp} = p_b\, C_{Lp}(1 + Z_v)Vd^{0.756Z_p - Z_v} \cdot N_M \qquad \text{...(143)}$$

where

$$N_M = \frac{\left(\frac{d}{11.24}\right)^{0.244Z_p}\left\{\left(\frac{d}{2.5}\right)^{1+Z_v-Z_p} - \left(\frac{d}{11.24}\right)^{1+Z_v-Z_p}\right\}}{1 + Z_v - Z_p} \qquad \text{...(144)}$$

(iii) UPPER ZONE ($d/2.5 < y_* \leqslant d$)

$$g_{SUp} = p_b\, C_{Lp}(1 + Z_v)Vd^{0.756Z_p - Z_v} \cdot N_U \qquad \text{...(145)}$$

where

$$N_U = \frac{\left(\frac{d}{11.24}\right)^{0.244Z_p}\left(\frac{d}{2.5}\right)^{0.5Z_p}\left\{d^{1+Z_v-1.5Z_p} - \left(\frac{d}{2.5}\right)^{1+Z_v-1.5Z_p}\right\}}{1 + Z_v - Z_p} \qquad \text{...(146)}$$

These functions derive from the expression:-

$$g_s = \int_{y_{*1}}^{y_{*2}} C_{xp} \; V \; dy_* \qquad \text{...(147)}$$

The bed load discharge is given by:-

$$g_{Sbp} = p[C_b \; V]_{y_*=2D}$$

$$= p_b \; C_{Lp}(1+Z_v)Vd^{0.756Z_p - Z_v}(2D)^{1+Z_v - 0.756Z_p} \qquad \text{...(148)}$$

The value of C_{Lp} is computed solving equations (141) and (142) with

$$g_{SLp} = GF_p \qquad \text{...(149)}$$

F Toffaleti suggests a check on the concentration at the level $y_* = 2D$ with the intention of, quite arbitrarily, avoiding unrealistically high values. The suggested maximum value is 100 lb/ft^3 or 1.6 tons/m^3,

$$\text{i.e. } (C_p)_{y_*=2D} = C_{Lp}\left(\frac{2D}{d}\right)^{-0.756Z_p} \leq 100 \text{ lb/ft}^3 \qquad \text{...(150)}$$

If condition (150) is not satisfied Toffaleti suggests making

$$C_{Lp} = \frac{100(\text{lb/ft}^3)}{(C_p)_{y_*=2D}} = \frac{1.6 \; (\text{tons/m}^3)}{(C_p)_{y_*=2D}} \qquad \text{...(151)}$$

The total sediment transport rate (bed material) in a size fraction, p, is given by

$$g_{stp} = g_{Sbp} + g_{SLp} + g_{SMp} + g_{SUp}$$

and the total sediment transport rate (bed material) for the whole bed material

$$g_{st} = \Sigma_1^p \; g_{stp} \qquad \text{...(153)}$$

For computational purposes equations (130) and (131) were brought to the following analytical forms:-

Denoting $$\overline{PAM} = \left\{ \left(\frac{10^5 \; g \; \nu}{32.2} \right)^{1/3} \Big/ \; 10 \; v_*' \right\} \qquad \text{...(154)}$$

If $0.13 < \overline{PAM} \leq 0.50$; $A_1 = 10 \; \overline{PAM}^{-1.487}$

If $0.50 < \overline{PAM} \leq 0.67$; $A_1 = 43 \; \overline{PAM}^{0.6142}$

If $0.67 < \overline{PAM} \leq 0.725$; $A_1 = 185 \; \overline{PAM}^{4.20}$

If $0.725 < \overline{PAM} \leq 1.25$; $A_1 = 49$

If $1.25 < \overline{PAM} \leq 10$; $A_1 = 24 \; \overline{PAM}^{2.79}$ ...(155)

Denoting $$\overline{FAC} = \overline{PAM} \cdot \frac{v_*^{\,2}}{gd} \cdot D_{65}(ft) \cdot 10^5 \qquad \text{...(156)}$$

If $\overline{FAC} \leq 0.25$; $k = 1$

If $0.25 < \overline{FAC} \leq 0.35$; $k = 5.37 \; \overline{FAC}^{1.248}$

If $0.35 < \overline{FAC} \leq 2.00$; $k = 0.50 \; \overline{FAC}^{-1.1}$ ...(157)

If the product $A_1 k < 16$, it must be taken arbitrarily as $A_1 k = 16$. v_*' is computed as in the Einstein Theory.

The computations proceed as follows:-

1. The shear velocity with respect to the grain, v_*', is computed as described herein for the Einstein methods using the relationships of Vanoni and Brooks (Ref 23).

2. The correction factors A_1 and k are determined using the sets of equations (155) and (157) or the original plots of Toffaleti, (Ref 22).

3. The exponent Z_v is computed using equation (129).

4. The exponent Z_p is evaluated for each fraction from equations (133), (134) and (135), the fall velocity being derived using equation A4 (see Appendix 1)

5. The nucleus load GF_p for each grain size fraction is evaluated using equation (139).

6. The value of the concentration C_{Lp} is determined from equation (149) and condition (150).

7. The bed load discharge and the suspension load within each zone and for each grain size fraction are computed using equations (141) to (151).

8. The total sediment transport rate (bed material) is then determined from equations (152) and (153).

TOTAL LOAD FORMULA OF P ACKERS AND W R WHITE (1972)

The general function of P Ackers and W R White (Refs 24 and 25) is one of the most recent formulae for the evaluation of the total sediment transport rate. It is based on physical considerations and on dimensional analysis. The various coefficients were derived using a wide range of flume data. The general function is:-

$$G_{gr} = C\left\{\frac{F_{gr}}{A} - 1\right\}^{m} \qquad \text{...(158)}$$

where

$$G_{gr}\ \text{(sediment transport)} = \frac{Xd}{sD}\left(\frac{v_*}{V}\right)^{n} \qquad \text{...(159)}$$

$$F_{gr}\ \text{(mobility)} = \frac{v_*^{\,n}}{\sqrt{gD(s-1)}}\left\{\frac{V}{\sqrt{32}\,\log\frac{10d}{D}}\right\}^{1-n} \qquad \text{...(160)}$$

m, C, A and n are given in terms of D_{gr}, the dimensionless particle size.

$$D_{gr} = D\left\{\frac{g(s-1)}{\nu^2}\right\}^{1/3} \qquad \text{...(161)}$$

For coarse sediments ($D_{gr} > 60$) these four coefficients are as follows:-

$$n = 0.0$$

$$A = 0.170$$

$$m = 1.50$$

$$C = 0.025 \qquad \ldots(162)$$

For the transitional sizes ($60 \geqslant D_{gr} \geqslant 1$):-

$$n = 1 - 0.56 \log D_{gr}$$

$$A = \frac{0.23}{\sqrt{D_{gr}}} + 0.14$$

$$m = \frac{9.66}{D_{gr}} + 1.34$$

$$\log C = 2.86 \log D_{gr} - (\log D_{gr})^{\frac{1}{2}} - 3.53 \qquad \ldots(163)$$

The particle mobility, F_{gr}, and the dimensionless grain size, D_{gr}, express the square root of the ratio shear forces/ immersed weight and the cube root of the ratio immersed weight/viscous forces respectively. Hence D_{gr} describes the influence of viscous forces on the sediment transport phenomenon. The sediment transport parameter, G_{gr}, is defined as:-

$$G_{gr} = \frac{\text{Shear forces}}{\text{Immersed weight}} \cdot E$$

where E is an efficiency factor "based on the work done in moving the material per unit time and the total stream power".

The coefficients n and A have physical meaning. The coefficient n relates to the transition zone of particle size; n = 1 for fine sediments ($D_{gr} = 1$) where correlation of F_{gr} and G_{gr} is with total shear, v_*; n = 0 for coarse sediments ($D_{gr} \geqslant 60$) where correlation of F_{gr} and G_{gr} is with grain shear, V and d/D being the representative variables.

At threshold conditions $G_{gr} = 0$ and equation (158) indicates that $F_{gr} = A$. Hence A is the critical mobility number. Comparison with Shields original parameters yields the relationship $Y_C = A^2$.

The general function of Ackers and White is based on flume data with sediment sizes in the range 0.04 mm < D < 4.0 mm. Correlation with lightweight sediments is good. A limitation on Froude number of ≤ 0.8 is suggested pending further investigations. For graded sediments the D_{35} of the bed material is suggested as the effective particle size. The equations are dimensionally homogeneous.

The calculation proceeds as follows:-

1. The value of D_{gr} is determined using equation (161).

2. The values of A, C, m and n associated with the computed D_{gr} value are determined using either the set of equations (162) or the set (163).

3. The value of the particle mobility number is evaluated using equation (160).

4. The value of G_{gr} is computed either using equation (158) or the graphs provided by White in Ref 25.

5. The sediment concentration by weight is determined from equation (159).

REFERENCES

1. SHIELDS A Anwendung der Aehnlichkeitsmechenik und der Turbulenzforschung auf die Geschiebebewegung. Mitteilungen der Preussichen Versuchsanstalt fur Wasserbau und Schiffbau, Helft 26, Berlin, 1936.

2. KALINSKE A A Movement of sediment as bed load in rivers. Am. Geophys. Union Trans., V. 28, pp 615-620, 1947.

3. LINDLEY E S Regime Channels. Proceedings, Punjab Engineering Congress, Vol. 17, 1919.

4. INGLIS CLAUDE C Meandèrs, and their bearing on river training. Maritime and Waterways Engineering Division, Institution of Civil Engineers, Paper No 7, 1947.

5. INGLIS CLAUDE C Historical note on empirical equations developed by Engineers in India for flow of water and sand in alluvial channel. Proc. 2nd Congress IAHR, Stockholm, June 1948.

6. MEYER-PETER E and MULLER R Formulae for bed load transport. Proc. 2nd Congress IAHR, Stockholm, June 1948.

7. EINSTEIN H A The bed load for sediment transportation in open channel flow. US Dept Agriculture, Soil Conservation, Ser. Tech. Bull. 1026, 1950.

8. KEULEGAN G H Laws of turbulent flow in open channels. National Bureau of Standards, Journal of Research, Vol 21, 1938, pp 701-741.

9. EINSTEIN H A and CHIEN N Transport of sediment mixtures with large ranges of grain size. Missouri River Division, Corps of Engineers Sediment Series No 2, University of California, June 1953.

10. BISHOP A A SIMONS D B and RICHARDSON E V Total bed material transport. Proc ASCE 91 HY2, February 1965.

11. ROUSE HUNTER (Editor) Engineering Hydraulics. Chapter XII, John Wiley and Sons, New York, 1950.

12. BAGNOLD R A The flow of cohesionless grain in fluids. Philosophical Trans. Roy. Soc. London, A, 249, No 964 (1956).

13. BAGNOLD R A An approach to the sediment transport problem from general physics. Geol. Survey, Professional Paper 422-I, US Government Printing Office, Washington, 1966.

14. LAURSEN E The total sediment load of stream. Journal of the Hydraulics Division, ASCE, Vol 54, No HY1, Proc. Paper 1530, February, 1958.

15. ROTTNER J A formula for bed load transportation. La Houille Blanche 4, No 3, 301-7, 1959.

16. YALIN M S An expression for bed load transportation. Proc. ASCE 89, HY3, 1963.

17. BLENCH T Mobile bed fluviology. Dept of Technical Services, University of Alberta, Alberta, Canada, 1966.

18. ENGELUND F and HANSEN E A monograph on sediment transport in alluvial streams. Teknisk Vorlag, Copenhagen, 1967.

19. ENGELUND F Hydraulics Resistance of alluvial streams. Journal of the Hydraulics Division, ASCE, Vol 92, No HY2, Proc Paper 4739, 1966.

20. GRAF W H Hydraulics of sediment transport. McGraw-Hill Book Company 1971.

21. TOFFALETI F B A procedure for computation of the total river sand discharge and detailed distribution, bed to surface. Committee on Channel Stabilisation, Corps of Engineers, US Army. Technical Report No 5, Vicksburg, Miss., November 1968.

22. TOFFALETI F B Definitive computations of sand discharge in rivers. Journal of Hydraulics Division, ASCE, Vol 95, No HY1, Proc. Paper 6357, 1969.

23. VANONI V A and BROOKS N H Laboratory studies of the roughness and suspended load of alluvial streams. Report E-68, Sedimentation Lab, California Inst. Tech., Pasadena, California, December 1957.

24. ACKERS P Sediment transport in channels: an alternative approach. Hydraulics Research Station, Report No INT 102, March 1972.

25. WHITE W R Sediment transport in channels: a general function. Hydraulics Research Station, Report No INT 104, June 1972.

26. RUBEY W W Settling velocities of gravel, sand and silt. Amer. Jour. Sci. 25, pp 325-338, 1933.

27. DUBOYS P Le Rhone et les rivieres a lit affouillable. Annales des Ponts et Chausees, 5 Ser., Vol 18, pp 141-195 (1879).

28. ROUSE HUNTER (Editor) Engineering Hydraulics. Chapter XII, John Wiley and Sons, New York 1950.

29. SHULITS SAMUEL The Shoklitsch bed load formula. Engineering, London. pp 644-646, June 21, 1935 and p 687, June 28, 1935.

30. MEYER-PETER E FARRE H and EINSTEIN A Nevere Versuchsresultate uber den Geschiebetrieb, Schweizerische Bauzeitung, Vol. 103, No 13, pp 147-150 (March 31, 1934).

31. CASEY H J Uber Geschiebebewegung. Doctoral Dissertation, Technische Hochschule, Berlin, Germany (1935). Translation No 35-1 US Army Waterways Experiments Station, Vicksburg, Miss.

32. STRAUB L G Missouri River Report, Appendix XV, US Serial No 9829, House Document No 238, 73rd Congress, 2nd Session (1935).

33. USWES Studies of river bed materials and their movement, with special reference to the lower Mississippi River, Paper 17 (1935).

34. HAYWOOD O G JR Flume experiments on the transportation by water of sands and light-weight materials. Dissertation, D. Sc., Mass. Inst. of Tech. (1940).

35. SCHOKLITSCHE A Handbuch des Wasserbaues, Springer-Verlag, Vienna, 2nd Ed., Vol. I (1950).

36. LEVI I I Dynamics of river beds. Leningrad, Moscow, 1948 (in Russian).

37. BOGARDI J L European concepts of sediment transportation. Journal of Hydraulics Division ASCE No HY1 January 1965.

38. ELZERMAN J J and FRIJLINK H C Present state of the investigation on bed load movement in Holland, IXth Assembly of the International Union of Geodesy and Geophysics, Brussels, No 1-4, pp 100-116 (1951).

39. COLBY B R and HEMBREE C H Computations of total sediment discharge, Niobrara River near Cody, Nebraska. US Geological Survey, Water Supply Paper No 1357, Washington, DC, 1955.

40. EGIAZAROFF I V Calculation of non-uniform sediment concentrations. Journal of the Hydraulics Division, ASCE, No HY4, July 1965.

41. BOGARDI J L Discussion of the total sediment load of streams by E M Laursen, ASCE, Vol 84, No HY6, Proc. Paper 1856, 1958.

42. GARDE R J and ALBERTSON M L Discussion of the total sediment load of streams. Proc. ASCE, Vol 84, No HY6.

43. COLBY B R Discharge of sands and mean velocity relationships in sand-bed streams. US Geological Survey, Prof. Paper 462-A, Washington, D.C., 1964.

44. WILSON K C Bed-load transport at high shear stress. Proc. ASCE 92, HY6 (November 1966).

45. CHANG F M SIMONS D B and RICHARDSON E V Total bed material discharge in alluvial channel. Proc. 12th Congress, IAHR, Vol I, September 1967.

46. SHULITS S and HILL R D JR Bed load formulas. Pennsylvania State University. December 1968.

APPENDICES

APPENDIX 1

Evaluation of Y_C, w and ν

Many of the theories included in this report contain the following three parameters in different combinations:-

1. The critical dimensionless parameter of Shields, of which the movement of the sediment starts, Y_C.

2. The fall velocity, w.

3. The kinematic viscosity, ν.

Any of these can be obtained from graphs displayed in the existing literature. However, for computational purposes it is necessary to express these sediment and fluid properties in a convenient analytical way.

1. <u>The Shields curve</u>

In the original paper of A A Shields (Ref 1) the initiation of motion is given as a relationship between the parameters

$$Y = \frac{v_*^2}{g(s-1)D} \quad \text{and} \quad R_* = \frac{v_* D}{\nu}$$

$$Y_C = f(R_*) \qquad \ldots(A1)$$

Here a more convenient form has been used eliminating the value v_* as a variable quantity, from the term of the right-hand side of equation (A1).

Putting $$D_{gr} = (R_*/Y)^{1/3} = D\left\{\frac{g(s-1)}{\nu^2}\right\}^{1/3} \qquad \ldots(A2)$$

the relationship (A1) can be rewritten as

$$Y_C = f(D_{gr}) \qquad \ldots(A3)$$

The Shields diagram, (Ref 1), as expressed above in equation (A3) was approximated by three curves with terminal points at $D_{gr} = 2$, 16 and 70.

If $\quad D_{gr} \leq 2.16$; $Y_C = 10^{-0.63-\log D_{gr}}$

If $2.16 \leq D_{gr} < 70$; $Y_C = 10^{0.0867(\log D_{gr}^3-1.82)^2-\log D_{gr}-0.60}$

If $\quad D_{gr} \geq 70$; $Y_C = 0.056$ ...(A4)

2. The fall velocity

There are various effects which complicate the uniform motion of a particle in a fluid, such as particle shape, multi-particle influence, turbulence, etc. There is no equation which describes the uniform motion of a single particle with complete precision. So, in order to compute analytically the settling velocity one is compelled to choose one of the several approximations to the problem. The Rubey (Ref 26) equation seems to be fairly reliable for particles with the shape of natural sands and it was used in this report to express the settling velocity. It is

$$w = F(s-1)^{\frac{1}{2}} g^{\frac{1}{2}} D^{\frac{1}{2}} \qquad \text{...(A5)}$$

where F is the dimensionless fall velocity and is given below as a function of D_{gr}, see equation (A2).

$$F = \left(\frac{2}{3} + \frac{36}{D_{gr}^3}\right)^{\frac{1}{2}} - \left(\frac{36}{D_{gr}^3}\right)^{\frac{1}{2}} \qquad \text{...(A6)}$$

3. The kinematic viscosity

In order to relate the kinematic viscosity to the temperature the well known expression of Poiseuille was used as follows:-

$$\nu = \frac{1.79\ 10^{-6}}{1 + 0.03368\ T_C + 0.000221\ T_C^2} \qquad \text{...(A7)}$$

in which ν is in m^2/s and the temperature, T_C, in degrees centigrade.

APPENDIX 2

Solution of integrals used in the Einstein methods

The bed load function and the total load function of H A Einstein include the integrals expressed in equations (33), (54) and (55) whose solution is somewhat cumbersome. The original work of Einstein, (Ref 7), gave special graphs to help in solving these integrals and readers interested should refer to them. Here a numerical integration is presented suitable for use in computer programs.

(i) <u>The expression (33) is of the form</u>:-

$$\int_a^b e^{-t^2} dt$$

which, expanding e^{-t^2} in Maclaurin's series and integrating term by term, becomes:-

$$\int_a^b e^{-t^2} dt = \int_a^b \left(1 - t^2 + \frac{t^4}{2!} - \frac{t^6}{3!} + \frac{t^8}{4!} + \ldots\right)$$

and so

$$\int_a^b e^{-t^2} dt = \left[t - \frac{t^3}{3} + \frac{t^5}{2!5} - \frac{t^7}{3!7} + \ldots + \frac{(-1)^{n-1} t^{2n-1}}{(n-1)!(2n-1)} + \ldots\right]_a^b \quad \ldots (A8)$$

The series in expression (A8) is convergent for any value of the variable t because in the limit $(n \rightarrow \infty)$

$$\frac{u_{n+1}}{u_n} = \frac{\dfrac{(-1)^n\ t^{2n+1}}{n!\ (2n+1)}}{\dfrac{(-1)^{n-1}\ t^{2n-1}}{(n-1)!(2n-1)}} = \frac{(-1)^n\ t^2(2n-1)}{(-1)^{n-1} n(2n+1)} = 0$$

(ii) <u>The integrals given in equations (54) and (55)</u>

are solved by Simpson's rule in which the curve represented by the function whose integration is sought is divided up into an even number of strips, m, of equal width and each portion approximated by a parabola. Thus the integral is evaluated as follows:-

$$I = \sum_{i=1}^{i=m} \frac{\Delta}{6} \left\{ f_{(t_{2i-1})} + 4f_{(t_{2i})} + f_{(t_{2i+1})} \right\} \qquad \ldots (A9)$$

where m = the number of strips

Δ = width of each strip

f(t) are:-

$$f^{i}(t) = \left(\frac{1-t}{t}\right)^{Z} \qquad \ldots (A10)$$

$$f^{ii}(t) = \left(\frac{1-t}{t}\right)^{Z} \ln t \qquad \ldots (A11)$$

with the integration limits equal to A and 1, see equations (54) and (55). Because of the particular nature of functions (A10) and (A11) it was found convenient to achieve the integration in two parts:

(i) Between t = A and $t + 10^{E}$ where E denotes the entire part of log A.

(ii) From $t = 10^{E}$ up to t = 1 making the computation in each logarithmic cycle in an independent form.

Thus the value of m, number of portions, is taken as a constant equal to 10 in each interval (or logarithmic cycle) and the variable t_i is computed at the extreme and middle points of each portion.

Then the function $f(t_i)$ is determined at the same points and the integral evaluated using formula (A9).

APPENDIX 3

Theories not evaluated

The following less widely used theories were not evaluated in the present exercise. The extra volume of work was not considered worthwhile.

Bed load equation of P DUBOIS (1879), Ref 27.

Bed load equation of A SCHOKLITSH (1934), Refs 28-29.

Bed load equation of F MEYER-PETER (1934), Ref 30.

Bed load equation of H J CASEY (1935), Ref 31.

Bed load equation of L G STRAUB (1935), Ref 32.

Bed load equation of WATERWAYS EXPERIMENT STATION (1935), Ref 33.

Bed load equation of O G HAYWOOD (1940), Ref 34.

Bed load equation of A SCHOKLITSH (1943), Ref 35.

Bed load equation of I I LEVY (1948), Refs 36-37.

Bed load equation of J J ELZERMAN and H C FRIJLINK (1951), Ref 38.

Total load equation of B R COLBY and C H HEMBREE (Modified Einstein Method) (1955), Ref 39.

Total load equation of I V EGIAZAROFF (1957), Ref 40.

Total load equation of J L BOGARDI (1958), Ref 41.

Total load equation of R J GARDE and M L ALBERTSON (1958), Ref 42.

Total load equation of B R COLBY (1964), Ref 43.

Total load equation of K C WILSON (1966), Ref 44.

Total load equation of F M CHANG, D B SIMONS and E V RICHARDSON (1967), Ref 45.

It should be noted that computations using the Modified Einstein method and the Colby method require the knowledge of the concentration of the suspended load. Many of the above formulae are summarised in Ref 46.

TABLE

TABLE 1

Summary of theoretical methods

Originator	Date	Type	Bed or total load	Graded sediments
A Shields	1936	Deterministic	Bed load	No
A A Kalinske	1947	Deterministic	Bed load	Yes
C Inglis	1947	Empirical	Total load	No
E Meyer-Peter and R Muller	1948	Deterministic	Bed load	No
H A Einstein	1950	Stochastic	Bed load	Yes
H A Einstein	1950	Stochastic	Total load	Yes
H A Einstein and C B Brown	1950	Stochastic	Bed load	No
A A Bishop, D B Simons and E V Richardson	1965	Stochastic	Total load	Yes
R A Bagnold	1956	Deterministic	Bed load	No
R A Bagnold	1966	Deterministic	Total load	No
E M Laursen	1958	Deterministic	Total load	Yes
J Rottner	1959	Deterministic	Bed load	No
M S Yalin	1963	Deterministic	Bed load	No
T Blench	1964	Empirical	Total load	No
F Engelund and E Hansen	1967	Deterministic	Total load	No
W H Graf	1968	Deterministic	Total load	No
F Toffaleti	1968	Deterministic	Total load	Yes
P Ackers and W R White	1972	Deterministic	Total load	No