A DIRECT METHOD OF CALCULATING
BOTTOM ORBITAL VELOCITY UNDER WAVES

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ABSTRACT

A method is presented for calculating the bottom orbital velocity under a wave simply and directly from its known height and period, and the water depth. When suitably nondimensionalised the results all fall on a single curve, with separate curves for monochromatic and random (JONSWAP spectrum) waves. The r.m.s velocity under random waves may be smaller or larger than that produced by a monochromatic wave of height $H_s$ and period $T_z$, depending on the water-depth. Direct methods of obtaining the effective period of the bottom velocity under random waves are also presented; these periods can be appreciably longer than $T_z$. 
TABLE 1: Values of non-dimensional bottom orbital velocity for monochromatic and random (JONSWAP spectrum) waves, and peak-period of bottom orbital velocity spectrum.

FIGURES

1. Dimensionless transfer functions as functions of dimensionless frequency. Monochromatic waves: $F_m = \frac{U_m}{2h/a^2g} = \frac{U_{rms}^2}{\frac{h}{H}}$ versus $x = \omega h/g$.
   Random waves (JONSWAP spectrum): $F_r = \left(\frac{H_s}{H}ight)^2 \frac{h}{g}$ versus $x_z = \frac{(2\pi)^2}{T_z} \frac{h}{g}$.

2. Bottom velocity for monochromatic waves ($U_m n /2H$ versus $T_m /T$) and random waves ($U_{rms} T_n /H_s$ versus $T_n /T_z$), where $T_n = (h/g)^{\frac{1}{2}}$.

3. Period $T_{pu}$ of the peak of the bottom orbital velocity spectrum.

APPENDIX - Relating monochromatic and random wave parameters
In many aspects of coastal engineering and oceanography it is necessary to know the orbital velocity at the sea bed produced by surface waves. Applications include sediment transport problems, forces on pipe-lines and structures at the sea bed, and the dissipation of wave energy. Frequently the bottom orbital velocity has to be deduced from surface measurements of wave height and period.

For monochromatic waves the appropriate quantity is the maximum bottom orbital velocity $U_m$ during the wave cycle. However, a naturally occurring random sea will have a broad spectrum of frequencies. Generally, information on the waves will be given in terms of the significant wave-height $H_s$ and the zero-crossing period $T_z$ (or the peak period $T_p$). It is tempting to assume that the sea can be represented by a monochromatic wave of height $H_s$ and period $T_z$ (or $T_p$). However, this may not be a good approximation as the attenuation of orbital velocity with depth depends strongly on wave period, so that the dominant waves at the bottom will have a period different to either $T_z$ or $T_p$. The near-bottom velocity cannot now be described by a single $U_m$, and it is usual to describe it by the standard deviation $U_{rms}$ of the time-series of instantaneous velocities. In some applications it is important to know the effective period of the orbital velocity, as well as the velocity itself.

The calculation of $U$ for monochromatic waves is not straightforward because it is necessary to solve the dispersion relation for the wave-number, which must be done graphically, iteratively, or as a series approximation. Calculation of $U_{rms}$ from a given surface elevation spectrum is considerably more laborious. The usual procedure is to convert the elevation spectrum to a bottom-velocity spectrum, which involves solving the dispersion relation at each frequency, and then integrating the resulting spectrum.
over the frequency range to yield $U_{\text{rms}}^2$. Calculation of the effective period is equally laborious.

The purpose of this report is to present a method of calculating $U_m$, $U_{\text{rms}}$, and the effective period, directly from the known quantities $H_s$ and $T_z$ together with the water depth. The results in each case are presented as single curves which are given in three alternative forms: graphically; as tabulated values; and as explicit algebraic expressions which approximate the curves closely.

2 MONOCHROMATIC WAVES

Consider a wave of amplitude $a = H/2$, and radian frequency $\omega = 2\pi/T$, where $H$ and $T$ are the wave height and period respectively, which gives rise to a maximum orbital velocity $U_m$ at the sea-bed (or, more correctly, just outside the thin wave boundary layer near the bed). Then $U_m$ is obtained using small-amplitude linear wave theory from

$$U_m = \frac{\omega}{a \sinh (kh)} \quad \frac{U_m}{H/2} = \frac{2\pi}{g m a h}$$

(1)

The wavenumber $k$ is related to the frequency $\omega$ by the dispersion relation

$$\omega^2 = gkh \tanh (kh),$$

(2)

where $g$ is the acceleration due to gravity and $h$ is the water depth. Define dimensionless variables:

$$x = \frac{\omega h}{g}$$

(3)

$$y = kh$$

(4)
Then Equation (1) becomes, after use of Equation (2),

$$F_m = \frac{U_m}{a^2 g}$$  \hspace{1cm} (5)

and the dispersion relation, Equation (2), becomes

$$x = y \tanh y.$$  \hspace{1cm} (6)

The dimensionless transfer function $F_m$ cannot be written explicitly in terms of $x$, and hence in terms of $H$ and $T$, because the dispersion relation, Equation (7), cannot be written explicitly as $y(x)$. However, as Equation (7) gives a one-to-one correspondence between $x$ and $y$, we see from Equation (6) that $F_m$ is a parametric function of $x$ alone. Both $F_m$ and $x$ contain only the known quantities $H$, $T$, $h$ and $g$, and the required quantity $U_m$. Thus a plot of $F_m$ versus $x$ (obtained by using $y$ as a parameter in Equations (6) and (7)) allows $U_m$ to be obtained directly from the known quantities (Fig 1). For small values of $x$ (shallow-water waves) the value of $F_m$ tends to one, and $F_m$ decreases monotonically with $x$ until it becomes very small for $x > 4$ (deep-water waves).

The quantities $F_m$ and $x$ are unnecessarily complicated for practical calculations, as they contain the squares of the quantities of interest and also contain some unnecessary constants. We therefore define more readily usable quantities by first introducing the natural scaling period $T_n$ defined by

$$T_n = \left(\frac{h}{g}\right)^{\frac{1}{2}}$$  \hspace{1cm} (8)

Then the required dimensionless quantities are
\[
\frac{U_{m,n}}{2H} \equiv \frac{F}{m} \tag{9}
\]

and

\[
\frac{T_n}{T} \equiv \frac{x}{2\pi} \tag{10}
\]

A plot of \( U_{m,n}/2H \) versus \( T_n/T \) (Fig 2) can be used directly for obtaining \( U_m \) from \( H, T, g \) and \( h \). For computer application, values of \( U_{m,n}/2H \) are tabulated against \( T_n/T \) in Table 1.

A 3-part explicit algebraic expression can be found which fits the curve in Figure 2 closely, as follows:

\[
F_{m} \equiv \frac{2U_{m}}{H} \left( \frac{h}{g} \right)^{\frac{1}{2}} = (1 - 0.670x + 0.110 x^2)^{\frac{1}{2}}, \ 0 < x < 1
\]

\[
= 1.72 x^{\frac{1}{2}} e^{-0.9529x}, \quad 1 < x < 3.2
\]

\[
= 2x^{\frac{1}{2}} e^{-x}, \quad 3.2 < x < \infty
\]

with

\[
x = \left( \frac{2\pi}{H} \right)^2 \frac{h}{g}
\]

Equation (11) fits the exact curve in Figure 2 to an accuracy of better than \( \pm 1\% \) over the entire range \( 0 < x < \infty \). The first and third parts of Equation (11) are based respectively on small and large argument approximations to \( \sinh y \) and \( \tanh y \) in Equations (6) and (7), together with some optimisation of the coefficients in the first part. Optimisation was also used to determine the coefficients in the middle part. The approximation given by Equation (11) has not been shown on Figure 2, because it is indistinguishable from the exact curve.
Under natural conditions the wave climate is represented by a spectrum of waves of different frequencies, amplitudes and directions. In many cases the only parameters which are known about the sea-conditions are the significant wave height $H_s$ and the zero-crossing period $T_z$. The best that can then be done is to fit a realistic surface elevation spectrum $S(\omega)$ to these two parameters. One of the most widely accepted two-parameter spectra is the JONSWAP spectrum (Hasselman et al, 1973), given by

$$S(\omega) = 2\pi\alpha g^2\omega^{-5} \exp\left\{-\frac{5}{4} \left(\frac{\omega}{\omega_p}\right)^4\right\} \varphi(\omega)$$

where

$$\varphi(\omega) = \exp\left\{-\frac{(\omega - \omega_p)^2}{2\beta^2\omega_p^2}\right\}.$$

Here $\omega_p$ is the radian frequency at the peak of the spectrum, $\gamma$ and $\beta$ are constants, and $\alpha$ is a variable which depends on the wind-speed and duration. We use the standard values of the constants, $\gamma = 3.3$ and $\beta = 0.07$ for $\omega > \omega_p$, $\beta = 0.09$ for $\omega < \omega_p$. The variables $\alpha_p$ and $\omega_p$ can be related to $H_s$ and $T_z$ respectively, so that a particular sea-state described only by $H_s$ and $T_z$ corresponds to a particular JONSWAP spectrum.

An additional complication of a random sea is that there is an appreciable spread in the wave directions, which is generally expressed by multiplying Equation (12) by a spreading function. For calculations of the wave energy dissipation rate the form of the spreading function can influence the dissipation rate by up to 20% (Brampton et al, 1984). However, because the bottom orbital velocity is related linearly to the surface elevation $\eta$, it is seen that the relationship between the quantities $U^{2}_{rms}$ and $H^{2}_{s}$ ($=16\sigma^{2}_{\eta}$) is independent of the spreading function.
The bottom velocity spectrum $S_u(\omega)$ is obtained by applying the dimensional transfer function given by Equation (1) to each frequency in the elevation spectrum:

$$S_u(\omega) = \frac{\omega^2}{\sinh^2(kh)} \cdot S_\eta(\omega)$$  \hspace{1cm} (13)

The variance of the bottom velocity is then obtained by integrating $S_u(\omega)$ over frequency

$$u^2_{\text{rms}} = \int_{0}^{\infty} S_u(\omega) \, d\omega$$  \hspace{1cm} (14)

We now define dimensionless variables analogous to those for monochromatic waves (Equations 3 and 5):

$$x = \left(\frac{2\pi}{T_z}\right)^2 \frac{h}{g}; \quad F_r = \left(\frac{U_{\text{rms}}}{H_s}\right)^2 \frac{h}{g}$$  \hspace{1cm} (15)

We note here that the standard deviation of the surface elevation of a random sea is $H_s/4$, and the standard deviations of the surface elevation and bottom velocity of a monochromatic wave are $H/\sqrt{2}$ and $U_m/\sqrt{2}$ respectively. Thus, to make analogous quantities for monochromatic and random waves correspond in meaning, we have introduced the factor 4 into the definition of $F_r$, and the factor 2 into the quantity $U^2_{\text{rms}}/2H$ defined earlier (see Appendix).

If we now further define $x_p = u^2_{\text{rms}}/g$, and write $\sinh^{-2}y = \text{func}_1(x)$ via Equation (7), then Equation (14) becomes

$$u^2_{\text{rms}} = \alpha \cdot g h \int_{0}^{\infty} x^{-3/2} \exp \left[ - \frac{5}{4} \left( \frac{x}{x_p} \right)^2 \right] \gamma(\psi(x)) \text{func}_1(x) \, dx$$  \hspace{1cm} (16)
Also, from the definition of $H_s$ in terms of the zeroth moment of the spectrum Equation (12), we have

$$
\left( \frac{H_s}{4} \right)^2 = \int_0^\infty S(\omega) \, d\omega
$$

$$
= \alpha h^2 \int_0^\infty x^{-5/2} \exp \left( -\frac{5}{2} \left( \frac{x}{\xi} \right)^2 \right) \psi \left( \frac{x}{\xi} \right) \, dx
$$

$$
= \alpha h^2 \text{func}_3 \left( \frac{x}{\xi} \right)
$$

(17)

From Equations (16) and (17) we obtain an expression for the dimensionless transfer function $F_r$ given by Equation (15):

$$
F_r = \frac{\text{func}_2 \left( \frac{x}{\xi} \right)}{\text{func}_3 \left( \frac{x}{\xi} \right)}
$$

(18)

Thus $F_r$ is a function of $x/\xi$ alone. By expressing $T_r^2$ as the ratio of the second and zeroth moments of $S(\omega)$, it can be related to the peak period $T_p$ of the JONSWAP spectrum given by Equation (12). With the values of $\beta$ and $\gamma$ given earlier,

$$
T_p = \frac{2\pi}{\omega} = 1.281 \, T_z
$$

(19)

Thus $x_z$ is proportional to $x_p$, and it follows that $F_r$ is a function only of $x_z$.

Values of $F_r$ for a range of values of $x_z$ have been calculated by performing numerically the integration given by Equation (14). An adaptation of a more general existing computer program described by Brampton et al (1984) was used, with an integration step of 0.1s and limits of the integration taken between periods of 0.1s and 5T_p. The resolution and limits are ample to give good accuracy. The resulting curve (Fig 1) follows the curve for monochromatic
waves for small values of \( x_z \), but becomes increasingly larger than it as \( x_z \) increases.

For simpler graphical use we have plotted \( \frac{U_{\text{rms}} T_n}{H_s} \) versus \( T_n/T_z \) (Fig 2) where \( T_n \) is defined by Equation (8). Values of \( U_{\text{rms}} T_n/H_s \) are tabulated against \( T_n/T_z \) in Table 1. For the random wave case it is not straightforward to obtain asymptotic expressions for \( F_r \) for small and large \( x_z \), as was done for monochromatic waves. Instead we have employed curve-fitting techniques to obtain an explicit algebraic expression which fits the curve in Figure 2 closely:

\[
\frac{U_{\text{rms}} T_n}{H_s} = \frac{0.25}{(1 + A t^2)^3}
\]

where

\[
A = \left[6500 + (0.56 + 15.54 t) 6\right]^{1/6} \tag{20}
\]

and

\[
t = \frac{T_n}{T_z} = \frac{1}{T_z} \left(\frac{h}{g}\right)^{1/2}
\]

Equation (20) fits the JONSWAP curve in Figure 2 to an accuracy of better than 1% in the range 0 < \( t < 0.55 \). Again we have not plotted Equation (20) on Figure 2 because it is indistinguishable from the exact curve.

We have calculated \( F_r \) only up to \( x_z = 11.8 \), ie. \( T_n/T_z = 0.55 \), because for larger values of \( x_z \) the bottom velocity is very small. For \( T_n/T_z = 0.55 \), Figure 2 gives \( U_{\text{rms}} T_n/H_s = 0.0038 \), so that

\[
U_{\text{rms}} = 0.0069 \frac{H_s}{T_z} \text{ at } T_n/T_z = 0.55 \tag{21}
\]

This provides an upper bound to velocities for \( T_n/T_z > 0.55 \). For example, if \( H_s = 4m \), \( T_z = 4s \), \( h = 47.5m \),
PERIOD OF BOTTOM ORBITAL VELOCITY

Although it is clear from the frequency dependence of the transfer function that the effective period of the bottom velocity will be larger than that of the wave elevations, it is less clear how the "effective period" should be defined. We examine here the period corresponding to the peak in the spectrum of bottom orbital velocity. This makes the largest contribution to the variance \( U_{\text{rms}}^2 \), which is the form in which wave-effects often appear in applications to sediment transport or to forces on structures near the sea-bed.

We therefore wish to compare the period \( T_{pu} \) at the peak of the velocity spectrum \( S_u(\omega) \) with the period \( T_p \equiv 2\pi/\omega \) at the peak of the elevation spectrum \( S_\eta(\omega) \). The maximum of \( S_u(\omega) \) is found by expressing Equation (13) in terms of \( F_m \) from Equations (1) and (5), and differentiating with respect to \( \omega \). After setting \( dS_u/d\omega = 0 \) and dividing through by \( S_u \), we obtain:

\[
\frac{1}{F_m} \frac{dF_m}{d\omega} = -\frac{1}{S_\eta} \frac{dS_\eta}{d\omega}
\]  \hspace{1cm} (22)

Making the substitution \( \phi = \omega/\omega_p \), Equation (22) becomes

\[
\frac{1}{F_m} \frac{dF_m}{dy} \frac{dy}{dx} \frac{dx}{d\omega} = -\frac{1}{S_\eta} \frac{dS_\eta}{d\phi} \frac{d\phi}{d\omega}
\]  \hspace{1cm} (23)

Substitution of \( F_m \) and \( dF_m/dy \) from Equation (6), \( dy/dx \) from Equation (7), \( dx/d\omega \) from Equation (3), \( S_\eta \) and \( dS_\eta/d\phi \) from Equation (12), and \( d\phi/d\omega = 1/\omega_p \), enables Equation (23) to be written in the form
\[ P(x) = Q(\phi) \]  

(24)

where

\[ P(x) = \frac{(2+4\gamma)e^{-4y} + 4y - 2}{1-e^{-4y} + 4y e^{-2y}} \]

and

\[ Q(\phi) = 5\phi^{-4} - 5 + \frac{\ln \gamma}{\beta^2} \phi (1-\phi) \exp \left\{-\frac{1}{2\beta^2} (1-\phi)^2\right\} \]

The functions \( Q(\phi) \) vs \( \phi \) and \( P(x) \) vs \( x \) (using \( y \) as parameter) can be plotted. Then if we denote by \( x_1 \) and \( \phi_1 \) the values of \( x \) and \( \phi \) which correspond to equal values of \( P(x) \) and \( Q(\phi) \), thereby satisfying Equation (21), we obtain, using the definition of \( \phi \),

\[ \frac{T_{pu}}{T_p} = \left(\frac{x}{x_1}\right)^{1/2} = \frac{1}{\phi_1} \]

and

(25)

\[ x_p = \frac{x}{\phi^2} \]

By picking off values of \( P(x) = Q(\phi) \), and making use of Equation (19), a plot of \( T_{pu}/T_z \) can be constructed (Fig 3). For small values of \( T_{n}/T_z \) (shallow-water waves) we find that \( T_{pu} \) tends to \( T_p \equiv 1.281 T_z \). As \( T_{n}/T_z \) increases, \( T_{pu}/T_z \) increases first slowly, then rather rapidly close to \( T_{n}/T_z = 0.4 \), and finally slowly again for \( T_{n}/T_z > 0.5 \), at which point \( T_{pu} \) is in excess of \( 1.7 T_z \).

Because the curve of \( T_{pu}/T_z \) vs \( T_{n}/T_z \) is not a simple shape we have not attempted to fit an algebraic approximation to it. Values of \( T_{pu}/T_z \) are tabulated against \( T_{n}/T_z \) in Table 1.
As illustrations consider a monochromatic wave of height $H = 5\text{m}$, period $T = 8\text{s}$, for two water-depths $h = 10\text{m}$ and $50\text{m}$.

For $h = 10\text{m}$, Eq (8) gives $T_n = 1.02\text{s}$, and hence $T_n/T = 0.127$. From the "monochromatic" curve in Fig 2 we obtain $U_m T_n/2H = 0.196$, and thus $U_m = 1.92\text{ms}^{-1}$.

For $h = 50\text{m}$ the corresponding values are $T_n = 2.26\text{s}$, $T_n/T = 0.282$, $U_m T_n/2H = 0.038$, and $U_m = 0.168\text{ms}^{-1}$.

Now consider a random sea having a JONSWAP spectrum with $H_s = 5\text{m}$ and $T_z = 8\text{s}$, in the same water-depths.

Then for $h = 10\text{m}$, Eq (8) gives $T_n = 1.02\text{s}$, and $T_n/T = 0.204$, and thus $U_{\text{rms}} = 1.00\text{ms}^{-1}$.

For $h = 50\text{m}$ the corresponding values are $T_n = 2.26\text{s}$, $T_n/T = 0.282$, $U_{\text{rms}} T_n/H_s = 0.087$, and $U_{\text{rms}} = 0.192\text{ms}^{-1}$.

In order to compare the random sea with a monochromatic wave of height $H_s$ and period $T_z$ it is first necessary to convert the velocity amplitude of the monochromatic wave to the corresponding root-mean-square value by $U_{\text{rms}} = U_m/\sqrt{2}$. Then for $h = 10\text{m}$ the random sea value $U_{\text{rms}} = 1.00\text{ms}^{-1}$ compares with the monochromatic wave value $U_{\text{rms}} = 1.36\text{ms}^{-1}$.

By contrast, for $h = 50\text{m}$ the random sea value $U_{\text{rms}} = 0.192\text{ms}^{-1}$ compares with the monochromatic wave value $U_{\text{rms}} = 0.119\text{ms}^{-1}$. Thus in shallow water an estimate of bottom orbital velocity based on a monochromatic wave of height $H_s$ and period $T_z$ will be a serious overestimate, but in deep water it will be a serious underestimate. The cross-over point, at which the monochromatic and random waves give the same $U_{\text{rms}}$, occurs at a depth given (using Fig 2) by $h = 0.049g T_z^2$. 

11
The peak-period of the bottom orbital velocity spectrum for the random sea with \( h = 10 \text{m} \), is obtained from Fig 3 with \( T_n/T_z = 0.127 \), for which
\[
T_{pu}/T_z = 1.296 \text{ giving } T_{pu} = 10.4 \text{s.}
\]
The corresponding value for \( h = 50 \text{m} \) is 10.6s. Neither value is very different from the peak-period of the surface elevation spectrum given by Eq (19) as \( T_p = 10.2 \text{s} \).

For a JONSWAP spectrum, which is relatively strongly peaked, \( T_p \) will be appreciably different from \( T_{pu} \) only for rather short period waves or rather deep water.

6 SUMMARY

Methods have been presented for calculating directly the bottom orbital-velocity and effective bottom period of waves of known height and period in water of depth \( h \). The results are presented graphically (for visual use), as tables (for computer application by look-up table), and as explicit algebraic expressions accurate to \( \pm 1\% \) (for use on pocket calculators and micro-computers). The methods can be summarised as follows:

1. Results, both for monochromatic and random waves, are scaled by the natural scale period \( T_n = (h/g)^{1/2} \).

2. For a monochromatic wave of height \( H \) and period \( T \) the amplitude \( U_m \) of the bottom orbital velocity can be obtained from the plot of \( U T_n/2H \) versus \( T_n/T\) given in Figure 2, or from Table 1 or Equation (11). These results can be used for laboratory as well as prototype waves.

3. For a random sea characterised by the significant wave height \( H_s \) and zero-crossing period \( T_z \), the root-mean-square bottom orbital velocity \( U_{rms} \) can be obtained from the plot of \( U T_n/H \) versus \( T_n/T_z \) given in Figure 2 or from Table 1 or Equation (20). This is based on a JONSWAP form for the elevation spectrum. Results are presented
for $0 < \frac{T_n}{T_z} < 0.55$ which covers the entire range of practical interest, but for values outside this range an upper bound to $U_{rms}$ is given by Equation (21).

4. The rms bottom orbital velocity calculated by assuming a JONSWAP spectrum (item 3 above) is larger or smaller than that calculated by assuming a monochromatic wave of height $H_s$ and period $T_z$ (item 2 above) depending on whether $h$ is larger or smaller than $0.049g T_z^2$ respectively. The difference may exceed 40% in either case.

5. The period $T_{pu}$ of the peak of the velocity spectrum is larger than the period $T_p$ of the elevation spectrum, by an amount which can be obtained from the plot of $T_{pu}/T_z$ versus $T_n/T_z$ given in Figure 3, or from Table 1. For many purposes the assumption that the effective period at the sea-bed is $T_p$ is adequate.

6. Where necessary, use the relation $T_p = 1.281 T_z$ throughout.

7 ACKNOWLEDGEMENTS

We are grateful to Mr M W Owen, Mr G Gilbert and Dr K R Dyer for their useful comments.

Table 1: Values of non-dimensional bottom orbital velocity for monochromatic and random (JONSWAP spectrum) waves, and peak-period of bottom orbital velocity spectrum.

\[ T_n = \sqrt{\frac{H}{g}} \]

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<tr>
<th>MONOCHROMATIC</th>
<th>JONSWAP SPECTRUM</th>
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Figures
Dimensionless transfer functions as functions of dimensionless frequency. Monochromatic waves: \( F_m = U_m \frac{2h}{a^2g} \) versus \( x = \frac{\omega^2 h}{g} \). Random waves (JONSWAP spectrum): \( F_r = \left( \frac{4U_{rms}}{H_s} \right)^2 \frac{h}{g} \) versus \( x_z = \left( \frac{2\pi}{T} \right)^2 \frac{h}{g} \).
Figure 2  Bottom velocity for monochromatic waves ($U_{\text{rms}} T_n/2H$ versus $T_n/T$) and random waves ($U_{\text{rms}} T_n/H_s$ versus $T_n/T z$), where $T_n = (h/g)^{1/2}$.
Figure 3 Period $T_{pu}$ of the peak of the bottom orbital velocity spectrum.

Short period waves for same $h$ \( \frac{T_n}{T_2} \) increases.
Appendix
Relating Monochromatic and Random Wave Parameters

Confusion can easily arise about the interrelationships of the various wave parameters in common use. We clarify here the relationships between the quantities used in this report.

For a sinusoidal monochromatic wave of height \( H \) and period \( T \), the angular frequency is:

\[
\omega = \frac{2\pi}{T},
\]

(A1)

the amplitude is

\[
a = \frac{H}{2}
\]

(A2)

and the variation of the surface elevation \( \eta(t) \) with time \( t \) at a particular point is given by:

\[
\eta = a \sin \omega t.
\]

(A3)

Thus the standard deviation \( \sigma_\eta \) of the surface is related to \( H \) by:

\[
H = 2\sqrt{2} \sigma_\eta
\]

(A4)

The bottom orbital velocity \( U(t) \) is given by

\[
U = U_m \sin \omega t
\]

(A5)

where \( U_m \) is the amplitude of the velocity. The standard deviation \( \sigma_u \) of the velocity, more commonly written as \( U_{\text{rms}} \), is thus:

\[
\sigma_u = U_{\text{rms}} = \frac{U_m}{\sqrt{2}}
\]

(A6)

For a random sea the significant wave height \( H_s \) is defined as:
H_s = 4 \sigma_\eta \quad \text{(A7)}

where \( \sigma_\eta \) is now the standard deviation of the random surface elevation \( \eta(t) \). (The rms wave height \( H_{\text{rms}} \) is also sometimes used, and is related to \( \sigma_\eta \) by \( H_{\text{rms}} = \sqrt{2 \sigma_\eta} \).) The term "random" is used to distinguish a naturally occurring multi-directional spectrum of waves from a unidirectional monochromatic sinusoidal wave, rather than in the usual statistical sense.

The period can be characterised either by the zero-crossing period \( T_z \) or by the angular frequency \( \omega_p \) at the peak of the surface elevation spectrum, leading to the peak period \( T_p \) given by:

\[
T_p = 2\pi/\omega_p \quad \text{(A8)}
\]

For any standard shape of spectrum (JONSWAP, Pierson-Moskowitz, etc) \( T_p \) is proportional to \( T_z \), with the constant of proportionality depending on the chosen spectral shape.

The root-mean-square bottom orbital velocity \( U_{\text{rms}} \) is related to the variance \( \sigma_u^2 \) of the random velocity vector \( u(t) \) by:

\[
U_{\text{rms}}^2 \equiv \sigma_u^2 = \left\langle u^2 \right\rangle \quad \text{(A9)}
\]

If a monochromatic wave of height \( H \) and a random sea of significant height \( H_s \) have the same variance \( \sigma_\eta^2 \) of the surface elevation, then, using eqs (A4) and (A7), they are related by:

\[
H = \frac{H_s}{\sqrt{2}} \quad \text{(A10)}
\]

If, instead, they have the same variance \( \sigma_u^2 \) of the bottom orbital velocity, then, using eqs (A6) and (A9), they are related by:
Thus, using eqs (A2), (A10) and (A11), the random wave quantity defined in eq (15):

\[ F_r = \left( \frac{4u_{\text{rms}}}{H} \right)^2 \left( \frac{h}{g} \right) \] 

(A12)
corresponds to the monochromatic quantity defined in eq (5):

\[ F_m = \frac{u_m}{a} \]

Similarly the monochromatic quantity \( u_{\text{rms}} \) corresponds to the random quantity \( u_{\text{rms}} T_n / H_s \).